# The fourth moment of automorphic L-functions at prime power level

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# Le quatrième moment des fonctions $L$ automorphes de niveau une grande puissance d'un nombre premier

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# Il momento quarto di funzioni automorfe $L$ a livello di grande potenza di un primo

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# The fourth moment of automorphic $L$-functions at prime power level

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**Abstract**

The main result of this dissertation is an asymptotic formula for the fourth moment of automorphic $L$-functions of prime power level $\rho^\nu$, $\nu \to \infty$. This is a continuation of the work of Rouymi, who computed the first three moments at prime power level, and a generalisation of results obtained for prime level by Duke, Friedlander & Iwaniec and Kowalski, Michel & Vanderkam.

**Résumé**

Le résultat principal de cette thèse est une formule asymptotique pour le quatrième moment des fonctions $L$ automorphes de niveau $\rho^\nu$, où $\rho$ est un nombre premier et $\nu \to \infty$. Il prolonge le travail de Rouymi, qui a calculé les trois premiers moments de niveau $\rho^\nu$, et il généralise les résultats obtenus en niveau premier par Duke, Friedlander & Iwaniec et Kowalski, Michel & Vanderkam.

**Sommario**

Il risultato principale di questa tesi è una formula asintotica per il momento quarto di funzioni automorfe $L$ a livello $\rho^\nu$, dove $\rho$ è un numero primo e $\nu \to \infty$. Questo estende il lavoro di Rouymi, che ha calcolato i primi tre momenti a livello $\rho^\nu$, e ciò generalizza i risultati per il livello primario di Duke, Friedlander & Iwaniec e Kowalski, Michel & Vanderkam.

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Symbols

| | |
|---|---|
| $\mu(n)$ | Möbius function |
| $\pi(n)$ | prime counting function |
| $\tau(n)$ | divisor function |
| $\tau_v(n)$ | $= \lvert n\rvert^{v-1/2} \sum_{d\mid n, d>0} d^{1-2v}$ |
| $\phi(n)$ | Euler's totient function |
| $\zeta(s)$ | Riemann's zeta function |
| $\xi(s)$ | Riemann's Xi function |
| $\zeta_q(s)$ | $= \zeta(s) \prod_{p\mid q}(1-p^{-s})$ |
| $\Gamma(s)$ | Gamma function |
| $\binom{n}{k}$ | binomial coefficient $= \frac{n!}{k!(n-k)!}$ |
| $\binom{n}{k_1, k_2, \ldots, k_m}$ | multinomial coefficient $= \frac{n!}{k_1!k_2!\ldots k_m!}$ |
| ${}_2F_1(a,b,c;1)$ | Gauss hypergemetric function $= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n$ |
| $\Delta(x_1, x_2, \ldots, x_n)$ | Vandermonde determinant $= \prod_{1\leqslant i<j\leqslant n}(x_i - x_j)$ |
| $J_n(x), Y_n(x), K_n(x)$ | Bessel functions |
| $k_0(x,v)$ | Bessel kernel $= \frac{1}{2\cos \pi v}(J_{2v-1}(x) - J_{1-2v}(x))$ |
| $k_1(x,v)$ | Bessel kernel $= \frac{2}{\pi} \sin \pi v K_{2v-1}(x)$ |
| $SL(2,\mathbb{Z})$ | modular group: $2 \times 2$ matrices with integral values and determinant 1 |
| $U(N)$ | unitary group: set of $N \times N$ matrices A such that $A^t\overline{A} = Id_N$ |
| $O(N)$ | orthogonal group: elements in $U(N)$ with real entries |
| $SO(N)$ | special orthogonal group: elements in $O(n)$ with determinant 1 |

| | |
|---|---|
| $USp(2N)$ | unitary symplectic group: elements $A \in U(2N)$ such that $AZ^tA = Z$ with $Z = \begin{pmatrix} 0 & Id_N \\ -Id_N & 0 \end{pmatrix}$ |
| $S(m,n,c)$ | Kloosterman sum defined in (2.1) |
| $e(x)$ | $= \exp(2\pi i x)$ |
| $\sum^h$ | harmonic average defined in (1.2) |
| $\sum_{q\mid c}$ | sum on $c \equiv 0(\text{mod q})$ |
| $(a,b)$ | the greatest common divisor of $a$ and $b$ |
| $[a,b]$ | the least common multiple of $a$ and $b$ |
| $\langle f,f\rangle_q$ | Petersson inner product defined in (2.12) |
| $a_n = o(b_n)$ | $a_n/b_n \to 0$ as $n \to \infty$ |
| $a_n = O(b_n)$ | there is $c > 0$ with $\lvert a_n\rvert < c\lvert b_n\rvert$ for all $n$ |
| $f \ll g$ | Vinogradov's symbol: $f = O(g)$ |
| $\hat{q}$ | $= \frac{\sqrt{q}}{2\pi}$ |
| $P(r)$ | polynomial dependence on parameter $r$ |
| $G(s)$ | even polynomial vanishing at all poles of $\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)$ in the range $\Re s \geqslant -L$ for some large constant $L > 0$. |

## Abbreviations

| | |
|---|---|
| RMT | Random Matrix Theory |

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# 1

# Introduction

Let $L(s,f)$ be an automorphic $L$-function associated to a primitive form $f$ of weight $k$ and level $q$. An important subject in analytic number theory is the behavior of such $L$-functions near the critical line $\Re s = 1/2$. Questions of particular interest are subconvexity bounds, equidistribution, gaps between zeros and proportion of vanishing (or non-vanishing) $L$-functions. See, for example, [9], [10], [11], [17], [24], [25], [34], [35], [37].

A possible way to analyse these problems is the method of moments and its variations: mollification and amplification. The given techniques proved to be extremely effective in the past years. However, the majority of results are known under assumption that the level $q$ is either prime or square-free number. See [9], [18], [23], [25], [40].

Recently, D. Rouymi considered the case $q = \rho^{\nu}$, where $\rho$ is a fixed prime number and $\nu \to \infty$. He computed the asymptotics of the first three moments and established a positive proportion of non-vanishing $L$-functions at the critical point $s = 1/2$.

Denote the $r$th harmonic moment by

$$M_r = \sum_{f \in H_k^*(q)}^{h} L(1/2, f)^r, \tag{1.1}$$

where

$$\sum_{f\in H_k^*(q)}^{h} := \sum_{f\in H_k^*(q)} \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f,f\rangle_q} \tag{1.2}$$

is the harmonic average over primitive newforms $H_k^*(q)$.

**Theorem 1.0.1.** *(Rouymi, [34]) Let $q=\rho^\nu$, $\nu\geqslant 3$. Then*

$$M_1 = \frac{\phi(q)}{q} + O_{k,\rho}(q^{-c}),\ 0<c<1/2,$$

$$M_2 = \left(\frac{\phi(q)}{q}\right)^2 \log q + O_{k,\rho}(1),$$

$$M_3 = \frac{1}{6}\left(\frac{\phi(q)}{q}\right)^4 (\log q)^3 + O_{k,\rho}((\log q)^2).$$

**Corollary 1.0.2.** *By Cauchy-Schwartz inequality*

$$\sum_{\substack{f\in H_k^*(q)\\ L(1/2,f)\neq 0}}^{h} 1 \geqslant \frac{(M_1)^2}{M_2} \gg \frac{1}{\log q}.$$

**Remark 1.0.3.** *Using the technique of mollification, Rouymi [35] obtained a bound independent of $\log q$. Let $k\geqslant 2$ be an even integer and $\rho$ be a prime number. Then for every $\delta>0$ there exists $\nu_0=\nu_0(k,\rho,\delta)$ such that for $\nu\geqslant\nu_0$ and $q=\rho^\nu$*

$$\sum_{\substack{f\in H_k^+(q)\\ L(1/2,f)\neq 0}}^{h} 1 \geqslant \frac{\rho-1}{6\rho} - \delta.$$

*Here $H_k^+(q)$ is a subset of $H_k^*(q)$ such that the sign of the functional equation* (2.23) *is plus.*

The fourth moment of automorphic $L$-functions of weight $k=2$ and prime level $q$, $q\to\infty$, was studied in [11] and [24].

**Theorem 1.0.4.** *(Kowalski, Michel and Vanderkam, [24], corollary 1.3) Let $q$ be a prime. For all $\epsilon > 0$*

$$\sum_{f \in H_2^*(q)}^{h} L(f, 1/2)^4 = Q(\log q) + O_\epsilon(q^{-1/12+\epsilon}), \tag{1.3}$$

*where $Q$ is a polynomial of degree 6 and leading coefficient is $\frac{1}{60\pi^2}$.*

In this dissertation, the result of theorem 1.0.4 is extended as follows.

- We consider the level of the form $q = \rho^\nu$, where $\rho$ is a fixed prime number and $\nu \to \infty$.
- We assume that the weight $k > 0$ is an arbitrary even integer.
- We slightly shift each $L$-function in the product from the critical line $\Re s = 1/2$

$$M_4(t_1, t_2, r_1, r_2) =$$

$$\sum_{f \in H_k^*(q)}^{h} L(1/2 + t_1 + ir_1, f)L(1/2 + t_1 - ir_1, f)L(1/2 + t_2 + ir_2, f)L(1/2 + t_2 - ir_2, f),$$

where $|t_1| < 1/2$, $|t_2| < 1/2$ and $t_1, t_2, r_1, r_2 \in \mathbb{R}$.

**Theorem 1.0.5.** *For all $\epsilon > 0$, the fourth moment can be written as follows*

$$M_4(t_1, t_2, r_1, r_2) = M^D + M^{OD} + M^{OOD} + O_{\epsilon,\rho,t_1,t_2}(P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2+\epsilon}(q^{-\frac{2k-3}{12}} + q^{-1/4})),$$

*where*

$$\begin{aligned} M^D + M^{OD} = \frac{\phi(q)}{q} &\sum_{\epsilon_1,\epsilon_2=\pm 1} \hat{q}^{-2t_1-2t_2+2\epsilon_1 t_1+2\epsilon_2 t_2} \\ &\times \zeta_q(1 + 2\epsilon_1 t_1)\zeta_q(1 + 2\epsilon_2 t_2)\frac{\prod \zeta_q(1 + \epsilon_1 t_1 + \epsilon_2 t_2 \pm ir_1 \pm ir_2)}{\zeta_q(2 + 2\epsilon_1 t_1 + 2\epsilon_2 t_2)} \\ &\times \frac{\Gamma(\epsilon_1 t_1 + ir_1 + k/2)\Gamma(\epsilon_1 t_1 - ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2)\Gamma(t_1 - ir_1 + k/2)}\frac{\Gamma(\epsilon_2 t_2 + ir_2 + k/2)\Gamma(\epsilon_2 t_2 - ir_2 + k/2)}{\Gamma(t_2 + ir_2 + k/2)\Gamma(t_2 - ir_2 + k/2)} \end{aligned} \tag{1.4}$$

*and*

$$M^{OOD} = \frac{\phi(q)}{q} \sum_{\epsilon_1,\epsilon_2=\pm 1} \hat{q}^{-2t_1-2t_2+2i\epsilon_1 r_1+2i\epsilon_2 r_2} \zeta_q(1+2i\epsilon_1 r_1)\zeta_q(1+2i\epsilon_2 r_2)$$

$$\times \frac{\prod \zeta_q(1\pm t_1 \pm t_2 + i\epsilon_1 r_1 + i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \frac{\Gamma(k/2-t_1+i\epsilon_1 r_1)\Gamma(k/2-t_2+i\epsilon_2 r_2)}{\Gamma(k/2+t_1-i\epsilon_1 r_1)\Gamma(k/2+t_2-i\epsilon_2 r_2)}. \quad (1.5)$$

The shifts simplify analysis of the off-off-diagonal term $M^{OOD}$, reveal more clearly a combinatorial structure of mean values and allow us to verify random matrix theory conjectures (including lower order terms) by Conrey, Farmer, Keating, Rubinstein and Snaith [7].

**Conjecture 1.0.6.** *(RMT, particular case of conjecture 3.0.6)*

*Up to an error term, we have*

$$M_4(t_1,t_2,r_1,r_2) = \frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2} \sum_{\substack{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4=\pm 1\\ \epsilon_1\epsilon_2\epsilon_3\epsilon_4=1}} \hat{q}^{t_1(\epsilon_1+\epsilon_2)+t_2(\epsilon_3+\epsilon_4)+ir_1(\epsilon_1-\epsilon_2)+ir_2(\epsilon_3-\epsilon_4)}$$

$$\times \left(\frac{\Gamma(-t_1-ir_1+k/2)\Gamma(-t_1+ir_1+k/2)\Gamma(-t_2-ir_2+k/2)\Gamma(-t_2+ir_2+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}\right)^{1/2}$$

$$\times \left(\frac{\Gamma(\epsilon_1(t_1+ir_1)+k/2)\Gamma(\epsilon_2(t_1-ir_1)+k/2)}{\Gamma(-\epsilon_1(t_1+ir_1)+k/2)\Gamma(-\epsilon_2(t_1-ir_1)+k/2)}\right)^{1/2}$$

$$\times \left(\frac{\Gamma(\epsilon_3(t_2+ir_2)+k/2)\Gamma(\epsilon_4(t_2-ir_2)+k/2)}{\Gamma(-\epsilon_3(t_2+ir_2)+k/2)\Gamma(-\epsilon_4(t_2-ir_2)+k/2)}\right)^{1/2}$$

$$\times \frac{\zeta_q(1+t_1(\epsilon_1+\epsilon_2)+ir_1(\epsilon_1-\epsilon_2))\zeta_q(1+t_2(\epsilon_3+\epsilon_4)+ir_2(\epsilon_3-\epsilon_4))}{\zeta_q(2+t_1(\epsilon_1+\epsilon_2)+t_2(\epsilon_3+\epsilon_4)+ir_1(\epsilon_1-\epsilon_2)+ir_2(\epsilon_3-\epsilon_4))}$$

$$\times \zeta_q(1+\epsilon_1(t_1+ir_1)+\epsilon_3(t_2+ir_2))\zeta_q(1+\epsilon_1(t_1+ir_1)+\epsilon_4(t_2-ir_2))$$

$$\times \zeta_q(1+\epsilon_2(t_1-ir_1)+\epsilon_3(t_2+ir_2))\zeta_q(1+\epsilon_2(t_1-ir_1)+\epsilon_4(t_2-ir_2)).$$

**Remark 1.0.7.** *The condition* $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4=\pm 1$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4=1$ *implies that there are eight terms in the sum. The four of them*

$$(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4) = (1,1,1,1),(1,1,-1,-1),(-1,-1,1,1),(-1,-1,-1,-1)$$

*coincide with the summands of* (1.4)*, and the other four*

$$(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)=(-1,1,-1,1),(-1,1,1,-1),(1,-1,-1,1),(1,-1,1,-1)$$

*with the summands of* (1.5)*.*

By letting the shifts tend to zero in theorem 1.0.5, we obtain an asymptotic formula for the fourth moment at the critical point $s=1/2$.

**Theorem 1.0.8.** *For all $\epsilon>0$, we have*

$$\sum_{f\in H_k^*(q)}^{h} L(1/2,f)^4 = Q(\log q)+O_{\epsilon,\rho}(q^{\epsilon}(q^{-\frac{2k-3}{12}}+q^{-1/4})), \tag{1.6}$$

*where $Q$ is a polynomial of degree* 6 *and leading coefficient is*

$$\left(\frac{\phi(q)}{q}\right)^7\frac{\rho^2}{\rho^2-1}\frac{1}{60\pi^2}. \tag{1.7}$$

The structure of the proof of theorem 1.0.5 is described by the figure 1.1. The main term of asymptotic formula consists of diagonal $M^D$, off-diagonal $M^{OD}$ and off-off-diagonal $M^{OOD}$ parts. Therefore, it requires three different stages of analysis.

First, we apply approximate functional equation (4.1.5) to the product of $L$-functions

$$L(1/2+t_1+ir_1,f)L(1/2+t_1-ir_1,f)L(1/2+t_2+ir_2,f)L(1/2+t_2-ir_2,f).$$

This allows us to use the Petersson trace formula (2.4.2). As a result, $M_4(t_1,t_2,r_1,r_2)$ splits into diagonal $M^D$ and non-diagonal $M^{ND}=M_1^{ND}+M_2^{ND}$ parts:

$$M^D=\frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2}\sum_{(q,n)=1}\frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n}W_{t_1,r_1}(\frac{n}{\hat{q}^2})W_{t_2,r_2}(\frac{n}{\hat{q}^2}), \tag{1.8}$$

$$M_1^{ND}=2\pi i^{-k}\hat{q}^{-2t_1-2t_2}\sum_{q|c}\frac{1}{c^2}T(c), \tag{1.9}$$

$$M_2^{ND}=-\frac{2\pi i^{-k}}{\rho}\hat{q}^{-2t_1-2t_2}\sum_{\frac{q}{\rho}|c}\frac{1}{c^2}T(c). \tag{1.10}$$

Here

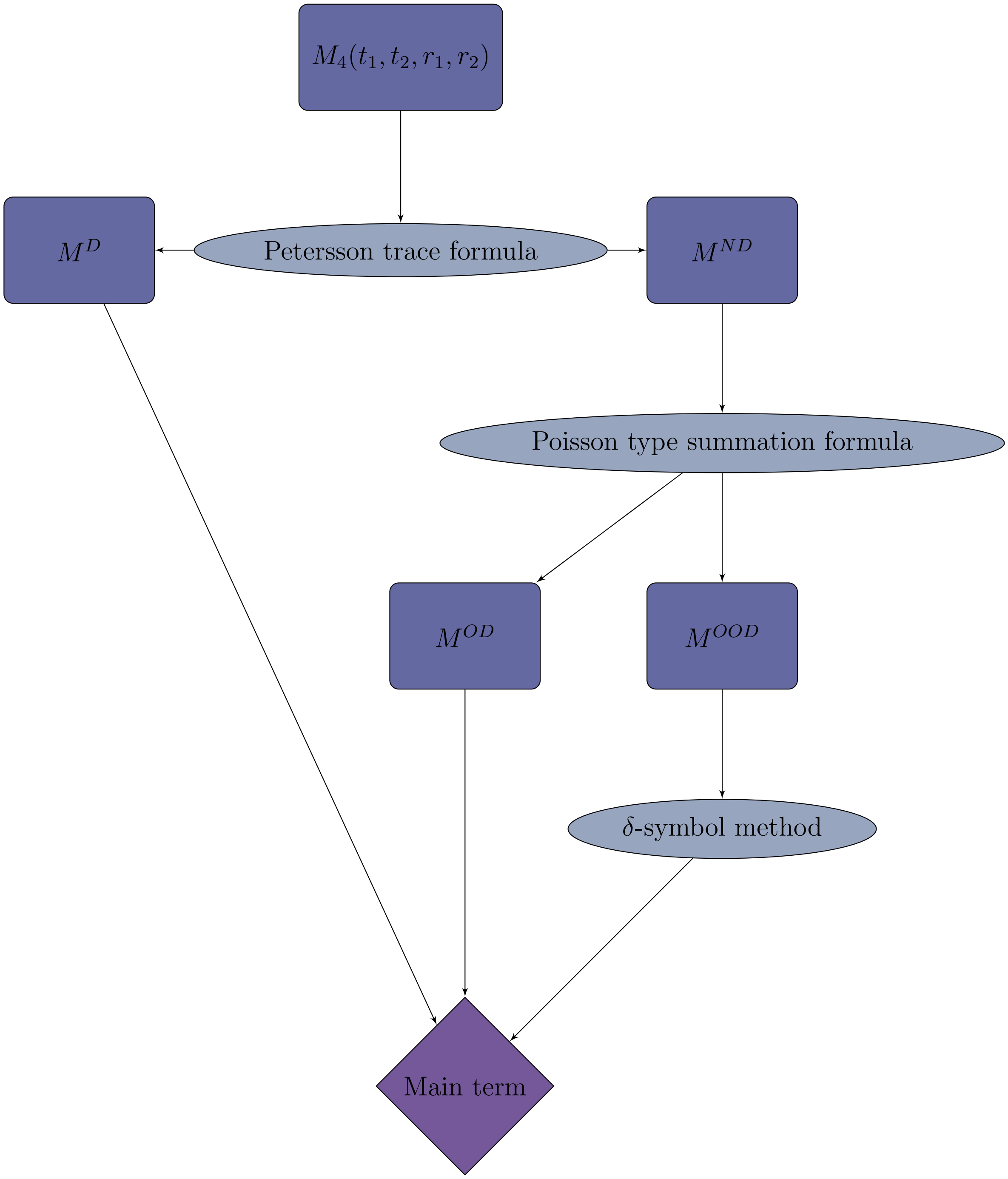
$$T(c) = c \sum_{\substack{m,n \\ (q,mn)=1}} \frac{\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)}{\sqrt{nm}} W_{t_1,r_1}(\frac{m}{\hat{q}^2}) W_{t_2,r_2}(\frac{n}{\hat{q}^2}) S(m,n,c) J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) \quad (1.11)$$

and

$$W_{t,r}(y) = \frac{1}{2\pi i}\int_{(3)} \frac{G(s)}{G(t)} \zeta_q(1+2s) \frac{\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} y^{-s} \frac{2sds}{s^2-t^2}. \quad (1.12)$$

The expression $T(c)$ includes sums of Kloosterman sums which we transform into Ramanujan sums using the Poisson summation formula connected with Eisenstein-Maass series (see theorem 4.5.4). Accordingly, the non-diagonal term is decomposed further into off-diagonal $M^{OD}$ and off-off-diagonal $M^{OOD}$ parts as shown in theorem 4.6.3. Asymptotics of $M^D + M^{OD}$ is given by theorem 4.0.4. The off-off-diagonal term $M^{OOD}$ is analysed using $\delta$-symbol method in chapter 5.

FIGURE 1.1: Structure of the fourth moment



# 2

# Background Information

## 2.1 Kloosterman sums

Consider the sum

$$S(m,n,c) = \sum_{\substack{d(\text{mod c})\\(c,d)=1}} e\left(\frac{m\bar{d}+nd}{c}\right) \tag{2.1}$$

with $d\bar{d} \equiv 1(\text{mod c})$.

It depends only on the residue class of $m$, $n$ modulo $c$ because $e^{2\pi ik} = 1$ for every $k \in \mathbb{Z}$. The value of $S(m,n,c)$ is always real because

$$\overline{S(m,n,c)} = S(m,n,c). \tag{2.2}$$

Further, since

$$\sum_{\substack{d(\text{mod c})\\(c,d)=1}} e\left(\frac{m\bar{d}+nd}{c}\right) = \sum_{\substack{e(\text{mod c})\\(c,e)=1}} e\left(\frac{me+n\bar{e}}{c}\right),$$

we have

$$S(m,n,c) = S(n,m,c), \tag{2.3}$$

$$S(ma,n,c) = S(m,na,c) \text{ if } (a,c)=1. \tag{2.4}$$

If we let one of the parameters $m$ or $n$ to be zero, then Kloosterman sum reduces to Ramanujan sum

$$S(0,n,c) = \sum_{\substack{d(\text{mod c})\\(c,d)=1}} e\left(\frac{nd}{c}\right). \tag{2.5}$$

Another important property is the twisted multiplicativity ([15] formula (4.12)). Suppose $(c_1, c_2) = 1$, $c_2\overline{c_2} \equiv 1(\text{mod } c_1)$, $c_1\overline{c_1} \equiv 1(\text{mod } c_2)$, then

$$S(m,n,c_1c_2) = S(m\overline{c_2}, n\overline{c_2}, c_1)S(m\overline{c_1}, n\overline{c_1}, c_2). \tag{2.6}$$

**Lemma 2.1.1.** *(Weil's bound, [42])*

$$|S(m,n,c)| \leqslant (m,n,c)^{1/2}c^{1/2}\tau(c). \tag{2.7}$$

**Lemma 2.1.2.** *(Royer, [36], Lemma A.12) Let $m,n,c$ be three strictly positive integers and $p$ be a prime number. Suppose $p^2$ divides $c$, $p$ divides $m$ and $p$ does not divide $n$, then $S(m,n,c) = 0$.*

## 2.2 Automorphic $L$-functions

Consider the Hecke congruence group

$$\Gamma_0(q) = \{\gamma \in SL(2,\mathbb{Z}) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, c \equiv 0 \pmod q\}. \tag{2.8}$$

It acts on the Poincaré upper-half plane $\mathbb{H} = \{z \in \mathbb{C}, \Im z > 0\}$ by linear fractional transformations

$$\gamma z = \frac{az+b}{cz+d}. \tag{2.9}$$

A holomorphic function $f$ on $\mathbb{H}$ is called a *cusp form* of weight $k$ and of level $q$ if it satisfies the following conditions:

$$f(\gamma z) = (cz+d)^k f(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q), \tag{2.10}$$

$$(\Im z)^{k/2}|f(z)| \text{ is bounded on } \mathbb{H}. \tag{2.11}$$

Let $S_k(q)$ be the space of cusp forms of weight $k \geqslant 2$ and of level $q$. It is equipped with the Petersson inner product

$$\langle f, g\rangle_q := \int_{F_0(q)} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}, \tag{2.12}$$

where $F_0(q)$ is a fundamental domain of the action of $\Gamma_0(q)$ on $\mathbb{H}$.

Any $f \in S_k(q)$ has a Fourier expansion at infinity

$$f(z) = \sum_{n \geqslant 1} a_f(n) e(nz). \tag{2.13}$$

According to the Atkin-Lehner theory [1], the space $S_k(q)$ can be decomposed into two subspaces

$$S_k(q) = S_k^{new}(q) \oplus S_k^{old}(q). \tag{2.14}$$

*The space of old forms* contains cusp forms of level $q$ coming from lower levels

$$S_k^{old}(q) = \left\{ f(lz) : lq'|q, q' < q, f(z) \in S_k(q') \right\}, \tag{2.15}$$

and *the space of new forms* is defined as an orthogonal compliment to $S_k^{old}(q)$.

We denote by $H_k^*(q)$ an orthogonal basis of $S_k^{new}(q)$. Elements of $H_k^*(q)$ with normalised Fourier coefficients

$$\lambda_f(n) := a_f(n) n^{-(k-1)/2}, \tag{2.16}$$

$$\lambda_f(1) = 1 \tag{2.17}$$

are called *primitive forms.*

Fourier coefficients of primitive forms satisfy the following properties

$$\lambda_f(n_1)\lambda_f(n_2) = \sum_{\substack{d|(n_1,n_2)\\(d,q)=1}} \lambda_f\left(\frac{n_1 n_2}{d^2}\right), \tag{2.18}$$

$$\lambda_f(n_1n_2) = \lambda_f(n_1)\lambda_f(n_2) \text{ if } (n_1, n_2) = 1, \tag{2.19}$$

$$\lambda_f(p^{j+1}) = \lambda_f(p)\lambda_f(p^j) - \lambda_f(p^{j-1}) \text{ for prime } p \text{ such that } (p, q) = 1. \tag{2.20}$$

Let $Re(s) > 1$, then for $f \in H_k^*(q)$ we define an *automorphic L-function*

$$L(s, f) = \sum_{n \geqslant 1} \lambda_f(n) n^{-s}. \tag{2.21}$$

*The completed L-function*

$$\Lambda(s, f) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s, f) \tag{2.22}$$

can be analytically continued on the whole complex plane and satisfies the functional equation

$$\Lambda(s, f) = \epsilon_f \Lambda(1 - s, f), \tag{2.23}$$

where $s \in \mathbb{C}$ and $\epsilon_f = \pm 1$.

## 2.3 Large sieve inequality

Suppose that $\lambda_1 = \lambda_1(q)$ is the smallest positive eigenvalue of the automorphic Laplacian for $\Gamma_0(q)$.

**Theorem 2.3.1.** *(Selberg's bound, [38]) One has that*

$$\lambda_1 \geqslant \frac{3}{16}. \tag{2.24}$$

Let $\theta_q := \sqrt{\max(0, 1 - 4\lambda_1)}$. We define

$$\|\mathbf{a}_M\|_2 = (\sum_{M < m \leqslant 2M} |a_m|^2)^{1/2} \tag{2.25}$$

and

$$\|\mathbf{b}_N\|_2 = (\sum_{N < n \leqslant 2N} |b_n|^2)^{1/2}. \tag{2.26}$$

**Theorem 2.3.2.** *(Deshouillers, Iwaniec, theorem* 9 *of [8]) Let* $r$ *and* $s$ *be positive coprime integers,* $C$, $M$, $N$ *be positive real numbers and* $g$ *be real-valued function of* $\mathfrak{C}^6$ *class (first and second derivatives are continuous for each of variables) with support in* $[M, 2M] \times [N, 2N] \times [C, 2C]$ *such that*

$$\left| \frac{\partial^{(j+k+l)}}{\partial m^{(j)} \partial n^{(k)} \partial c^{(l)}} g(m, n, c) \right| \leqslant M^{-j} N^{-k} C^{-l} \text{ for } 0 \leqslant j, k, l \leqslant 2. \tag{2.27}$$

*Then for any* $\epsilon > 0$ *and complex sequences* $\mathbf{a}$, $\mathbf{b}$ *one has*

$$\sum_{(c,r)=1} \sum_m a_m \sum_n b_n g(m, n, c) S(m\bar{r}, \pm n, sc) \ll_\epsilon C^\epsilon \left(1 + \frac{s\sqrt{rC}}{\sqrt{MN}}\right)^{\theta_{rs}}$$

$$\times \frac{(s\sqrt{r}C + \sqrt{MN} + \sqrt{sMC})(s\sqrt{r}C + \sqrt{MN} + \sqrt{sNC})}{s\sqrt{r}C + \sqrt{MN}} \|\mathbf{a}_M\|_2 \|\mathbf{b}_N\|_2. \tag{2.28}$$

## 2.4 Petersson trace formula in case of prime power level

The key ingredient of our proof is the Petersson trace formula, which allows to express Fourier coefficients of cusp forms in terms of Kloosterman sums weighted by Bessel functions.

**Theorem 2.4.1.** *(Petersson trace formula) For* $m, n \geqslant 1$ *we have*

$$\Delta_q(m, n) := \sum_{f \in H_k(q)}^{h} \lambda_f(m) \lambda_f(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{q|c} \frac{S(m, n, c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{2.29}$$

If $q$ is a prime number and $k < 12$, the Petersson trace formula also works for moments of $L$–functions associated to primitive forms since the space of old forms is empty and $H_k^*(q) = H_k(q)$ .

When $q$ is a power of prime, one needs to exclude the contribution of old forms. This has been done by Rouymi. He constructed a special basis in order to find an analogue of the Petersson trace formula for primitive forms at prime power level.

**Theorem 2.4.2.** *(Rouymi, remark* 4 *of [34])*

*Let* $q = \rho^\nu$, $\nu \geqslant 2$, *then*

$$\Delta_q^*(m,n) := \sum_{f \in H_k^*(q)}^{h} \lambda_f(m)\lambda_f(n) = \begin{cases} \Delta_q(m,n) - \frac{\Delta_{q/\rho}(m,n)}{\rho - \rho^{-1}} & \text{if } (q,mn) = 1 \text{ and } \nu = 2, \\ \Delta_q(m,n) - \frac{\Delta_{q/\rho}(m,n)}{\rho} & \text{if } (q,mn) = 1 \text{ and } \nu \geqslant 3, \\ 0 & \text{if } (q,mn) > 1. \end{cases} \tag{2.30}$$

## 2.5 Random matrix theory and moments of automorphic $L$-functions

The behaviour of mean values of $L$-functions at the critical point can be modelled using characteristic polynomials of random matrices for compact groups $O(N)$, $USp(2N)$ and $U(N)$ . Accordingly, we distinguish families of $L$-functions with othogonal, symplectic and unitary symmetry types. The most general prediction is given in [6].

**Conjecture 2.5.1.** *(Conrey, Farmer) Suppose the family of $L$-functions is partially ordered by a conductor $c(f)$ and $\mathcal{Q}^*$ is the number of elements with $c(f) \leqslant \mathcal{Q}$. Then*

$$\frac{1}{\mathcal{Q}^*} \sum_{\substack{f \in F \\ c(f) \leqslant \mathcal{Q}}} V(L(1/2, f))^n \sim \frac{g_n a_n}{\Gamma(1 + B(n))} (\log \mathcal{Q}^A)^{B(n)}. \tag{2.31}$$

*Here $V(z) = |z|^2$ for unitary symmetry and $V(z) = z$ for othogonal and symplectic symmetries. The constant $A$ depends both on the type of symmetry and the functional equation. The values of $g_n$, $B(n)$ are completely determined by the symmetry type and $a_n$ can be computed for each particular family.*

Automorphic $L$-functions for primitive forms is an example of family with orthogonal symmetry type. In that case, $V(z) = z$ and $B(n) = 1/2n(n-1)$. There are two categories of $L$-functions of this type: even and odd, corresponding to $SO(2N)$ and $SO(2N+1)$, respectively.

**Conjecture 2.5.2.** *(Keating, Snaith [22]) Let $q$ be a prime number. Then*

$$\sum_{f \in H_2(q)} L_f(1/2, f)^n = \frac{q}{3} R_n(\log q) + O(q^{1/2+\epsilon}),$$

*where $R_n$ is a polynomial of degree $\frac{1}{2}n(n-1)$ with leading coefficient $\frac{a_n g_n}{(n(n-1)/2)!}$. Here*

$$a_n = \prod_{p \nmid q} (1-1/p)^{n(n-1/2)} \frac{2}{\pi} \int_0^\pi \sin^2\theta \left( \frac{e^{i\theta}(1-e^{i\theta}/\sqrt{p})^{-1} - e^{-i\theta}(1-e^{-i\theta}/\sqrt{p})^{-1}}{e^{i\theta} - e^{-i\theta}} \right)^n d\theta,$$

$$g_n = 2^{n-1}(n(n-1)/2)! \prod_{j=1}^{n-1} \frac{j!}{(2j)!}.$$

**Remark 2.5.3.** *This conjecture has been proven for $n = 1, 2, 3, 4$. See [9], [17], [24], [25].*

A more general conjecture, which describes not only the leading term but also all lower order terms, is given in [7].

**Conjecture 2.5.4.** *(Conrey, Farmer, Keating, Rubinstein, Snaith) Let $q$ be square-free and let*

$$X(s) := \left(\frac{q}{4\pi^2}\right)^{1/2-s} \frac{\Gamma(1/2 - s + k/2)}{\Gamma(-1/2 + s + k/2)}.$$

*Then*

$$\sum_{f \in H_k^*(q)} \langle f, f \rangle_q^{-1} L(1/2+\alpha_1, f) L(1/2+\alpha_2, f) \dots L(1/2+\alpha_r, f)$$

$$= \prod_{j=1}^r X(1/2-\alpha_j)^{-1/2} \sum_{\substack{\epsilon_j = \pm 1 \\ \prod_{j=1}^r \epsilon_j = 1}} X(1/2+\epsilon_j\alpha_j)^{-1/2}$$

$$\times \prod_{1 \leqslant i < j \leqslant r} \zeta(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j) A_r(\epsilon_1\alpha_1, \dots, \epsilon_r\alpha_r)(1+O(kq)^{-1/2+\epsilon}),$$

*where $A_r$ is absolutely convergent for $\Re z_j < 1/2$ and it is given by*

$$A_r(z_1, z_2, \dots, z_r) = \prod_{p \nmid q} \prod_{1 \leqslant i < j \leqslant r} \left(1 - \frac{1}{p^{1+z_i+z_j}}\right)$$

$$\times \frac{2}{\pi} \int_0^\pi \sin^2\theta \prod_{j=1}^r \frac{e^{i\theta}(1-e^{i\theta}/p^{1/2+z_j})^{-1} - e^{-i\theta}(1-e^{-i\theta}/p^{1/2+z_j})^{-1}}{e^{i\theta}-e^{-i\theta}} d\theta.$$

# 3

# Random matrix theory and heuristic predictions in case of prime power level

Conjecture 2.5.4 gives predictions for moments of automorphic $L$-functions of square-free level. Following the recepe described in [7], we find a similar conjecture in case of prime power level $q = \rho^\nu$, $\nu \geqslant 2$.

Let $\hat{q} = \frac{\sqrt{q}}{2\pi}$. The functional equation (2.23) can be written as

$$L(s, f) = \epsilon_f X_f(s) L(1 - s, f), \tag{3.1}$$

where

$$X_f(s) := \hat{q}^{1-2s} \frac{\Gamma(1/2 - s + k/2)}{\Gamma(-1/2 + s + k/2)}. \tag{3.2}$$

We denote the Vandermonde determinant by

$$\Delta(z_1, \ldots, z_r) := \prod_{1 \leqslant i < j \leqslant r} (z_j - z_i). \tag{3.3}$$

Suppose $k$ be an even integer, $q = \rho^\nu$, $\nu \geqslant 2$ and

$$C(q) := \begin{cases} \frac{\phi(q)}{q} \text{ if } \nu \geqslant 3, \\ \frac{\rho^2 - \rho - 1}{\rho^2 - 1} \text{ if } \nu = 2 \end{cases}.$$

Let

$$A_r(z_1,\ldots,z_r) := \prod_{p\nmid q}\prod_{1\leqslant i<j\leqslant r}\left(1-\frac{1}{p^{1+z_i+z_j}}\right)$$
$$\times \frac{2}{\pi}\int_0^\pi \sin^2\theta \prod_{j=1}^r \frac{e^{i\theta}\left(1-\frac{e^{i\theta}}{p^{1/2+z_j}}\right)^{-1} - e^{-i\theta}\left(1-\frac{e^{-i\theta}}{p^{1/2+z_j}}\right)^{-1}}{e^{i\theta}-e^{-i\theta}}d\theta \quad (3.4)$$

and

$$T(z_1,\ldots,z_r) := A_r(z_1,z_2,\ldots,z_r)\prod_{1\leqslant i<j\leqslant r}\zeta_q(1+z_i+z_j). \quad (3.5)$$

**Proposition 3.0.5.** *For the fourth moment*

$$A_r(z_1,z_2,z_3,z_4) = \frac{1}{\zeta_q(2+z_1+z_2+z_3+z_4)}. \quad (3.6)$$

*Proof.* Consider

$$I(z_1,\ldots,z_4) := \frac{2}{\pi}\int_0^\pi \sin^2\theta \prod_{j=1}^4 \frac{e^{i\theta}\left(1-\frac{e^{i\theta}}{p^{1/2+z_j}}\right)^{-1} - e^{-i\theta}\left(1-\frac{e^{-i\theta}}{p^{1/2+z_j}}\right)^{-1}}{e^{i\theta}-e^{-i\theta}}d\theta$$
$$= \frac{2}{\pi}\int_0^\pi \sin^2\theta \prod_{j=1}^4 \frac{p^{1+2z_j}}{p^{1+2z_j}-2p^{1/2+z_j}\cos\theta+1}d\theta.$$

We note that

$$\frac{2}{\pi}\int_0^\pi \frac{\sin^2\theta}{(a^2-2a\cos\theta+1)(b^2-2b\cos\theta+1)(c^2-2c\cos\theta+1)(d^2-2d\cos\theta+1)}d\theta$$
$$= \frac{abcd-1}{(ab-1)(ac-1)(ad-1)(bc-1)(bd-1)(cd-1)}.$$

Therefore,

$$I(z_1,\ldots,z_4) = \frac{p^{4+2z_1+2z_2+2z_3+2z_4}(p^{2+z_1+z_2+z_3+z_4}-1)}{\prod_{1\leqslant i<j\leqslant 4}(p^{1+z_i+z_j}-1)}$$

and

$$A_r(z_1,z_2,z_3,z_4) = \prod_{p\nmid q} p^{4+2z_1+2z_2+2z_3+2z_4}(p^{2+z_1+z_2+z_3+z_4}-1)\prod_{1\leqslant i<j\leqslant 4}\frac{1}{p^{1+z_i+z_j}}$$
$$= \prod_{p\nmid q}\left(1-\frac{1}{p^{2+z_1+z_2+z_3+z_4}}\right) = \frac{1}{\zeta_q(2+z_1+z_2+z_3+z_4)}.$$

□

Consider a product of $r$ shifted $L$-functions

$$L(s,\alpha_1,\alpha_2,\ldots,\alpha_r) := L(s+\alpha_1,f)L(s+\alpha_2,f)\ldots L(s+\alpha_r,f). \tag{3.7}$$

In section 3.3.1 we obtain the following conjecture.

**Conjecture 3.0.6.**

$$\sum_{f\in H_k^*(q)}^{h} L(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) = M(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) + error, \tag{3.8}$$

*where*

$$M(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) = C(q)\prod_{j=1}^{r} X_f(1/2-\alpha_j)^{-1/2}$$
$$\times \sum_{\substack{\epsilon_j=\pm1\\ \prod_{j=1}^{r}\epsilon_j=1}}\prod_{j=1}^{r} X_f(1/2+\epsilon_j\alpha_j)^{-1/2}T(\epsilon_1\alpha_1,\ldots,\epsilon_r\alpha_r).$$

Conjecture 3.0.6 can be stated in terms of contour integrals for odd ($\epsilon_f=-1$) and even ($\epsilon_f=+1$) forms separately.

**Conjecture 3.0.7.**

$$\sum_{\substack{f\in H_k^*(q)\\ f \text{ is even}}}^{h} L(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) = 1/2M_1(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) + error,$$

*where*

$$M_1(1/2,\alpha_1,\alpha_2,\dots,\alpha_r)=C(q)\frac{(-1)^{r(r-1)/2}}{(2\pi i)^r}\frac{2^r}{r!}\prod_{j=1}^{r}X_f(1/2-\alpha_j)^{-1/2}$$

$$\times\oint\dots\oint\prod_{j=1}^{r}X_f(1/2+z_j)^{-1/2}T(z_1,\dots,z_r)\frac{\Delta(z_1^2,\dots,z_r^2)\prod_{j=1}^{r}z_j}{\prod_{i=1}^{r}\prod_{j=1}^{r}(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\dots dz_r.$$

**Conjecture 3.0.8.**

$$\sum_{\substack{f\in H_k^*(q)\\ f\ is\ odd}}^{h}L(1/2,\alpha_1,\alpha_2,\dots,\alpha_r)=1/2M_2(1/2,\alpha_1,\alpha_2,\dots,\alpha_r)+error,$$

*where*

$$M_2(1/2,\alpha_1,\alpha_2,\dots,\alpha_r)=C(q)\frac{(-1)^{r(r-1)/2}}{(2\pi i)^r}\frac{2^r}{r!}\prod_{j=1}^{r}X_f(1/2-\alpha_j)^{-1/2}$$

$$\times\oint\dots\oint\prod_{j=1}^{r}X_f(1/2+z_j)^{-1/2}T(z_1,\dots,z_r)\frac{\Delta(z_1^2,\dots,z_r^2)\prod_{j=1}^{r}\alpha_j}{\prod_{i=1}^{r}\prod_{j=1}^{r}(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\dots dz_r.$$

**Remark 3.0.9.** *These results are consistent with asymptotic formulas for mean values of characteristic polynomials of odd and even othogonal matrices. See Theorem 1.5.6 of [7].*

## 3.1 Averages over the family

*Log conductor* of $f$ is defined as

$$c(f):=|(\epsilon_f X_f)'(1/2)|. \tag{3.9}$$

Equation (3.2) implies that

$$c(f)=2\log\hat{q}+2\frac{\Gamma'}{\Gamma}(k/2). \tag{3.10}$$

The number of elements for which log conductor doesn't exceed $T$ is called *counting function*

$$M(T):=\sharp\{f:c(f)\leqslant T\}. \tag{3.11}$$

Let $G$ be a function defined on the family $F$. Then its *expected value* is

$$\langle G(f)\rangle = \lim_{T\to\infty} M(T)^{-1} \sum_{\substack{f\in F\\ c(f)\leqslant T}} G(f). \tag{3.12}$$

Here we consider harmonic average with respect to Petersson inner product (2.12)

$$\sum_{f\in H_k^*(q)}^{h} := \sum_{f\in H_k^*(q)} \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f,f\rangle_q}. \tag{3.13}$$

## 3.2 Mean values of Fourier coefficients

The property of multiplicity (2.18) implies that

$$\lambda_f(n_1)\lambda_f(n_2)\dots\lambda_f(n_r) = \sum_{j\geqslant 1} b_j\lambda_f(j) \tag{3.14}$$

for some $b_j$.

**Lemma 3.2.1.** *Let $q=\rho^\nu$, $(q, n_1n_2\dots n_r)=1$. Then*

$$\delta(n_1,n_2,\dots,n_r) := \langle\lambda_f(n_1)\lambda_f(n_2)\dots\lambda_f(n_r)\rangle = b_1C(q). \tag{3.15}$$

*Proof.* It follows from equation (3.14) that

$$\delta(n_1,n_2,\dots,n_r) = \lim_{q\to\infty}\sum_{j\geqslant 1} b_j \sum_{f\in H_k^*(q)}^{h} \lambda_f(j)\lambda_f(1).$$

Weil's bound (2.7) and asymptotic formula (C.7) imply that

$$\sum_{c=1}^{\infty} \frac{S(j,1,cq)J_{k-1}\left(\frac{4\pi\sqrt{j}}{cq}\right)}{cq} \ll_{k,j} \sum_{c=1}^{\infty}\tau(cq)(cq)^{-k+1/2}$$

tends to zero as $q\to\infty$.

Applying Petersson's trace formula (Theorem 2.4.2), we have

$$\delta(n_1,n_2,\dots,n_r) = \lim_{q\to\infty}\sum_{j\geqslant 1} b_j\Delta_q^*(j,1) = b_1C(q).$$

□

**Lemma 3.2.2.** *If $p \nmid q$, then*

$$\delta(p^{t_1}, \ldots, p^{t_r}) = c_0 C(q)$$

*and*

$$c_0 = \frac{2}{\pi} \int_0^{\pi} \sin^2\theta \prod_{j=1}^{r} \frac{e^{i(t_j+1)\theta} - e^{-i(t_j+1)\theta}}{e^{i\theta} - e^{-i\theta}} d\theta. \tag{3.16}$$

*If $p|q$ and $t_1, t_2, \ldots, t_r \neq 0$, then*

$$\delta(p^{t_1}, \ldots, p^{t_r}) = 0.$$

*Proof.* Fourier coefficients $\lambda_f(p^j)$ satisfy the same recurrence relation as Chebyshev polynomials of the second kind (compare (2.20) and (F.4)). Therefore,

$$\lambda_f(p^j) = U_j(\cos\theta_{f,p}).$$

Consider

$$U_{t_1} U_{t_2} \ldots U_{t_r} = \sum_{l \geqslant 0} c_l U_l.$$

On the one hand, Lemma 3.2.1 gives

$$\delta(p^{t_1}, \ldots, p^{t_r}) = c_0 C(q).$$

On the other hand, the property of orthogonality (F.7) implies that

$$c_0 = \frac{2}{\pi} \int_0^{\pi} \sin^2\theta \prod_{j=1}^{r} \frac{e^{i(t_j+1)\theta} - e^{-i(t_j+1)\theta}}{e^{i\theta} - e^{-i\theta}} d\theta.$$

□

**Corollary 3.2.3.** *Assume that $p \nmid q$. Then*

$$\sum_{t_1, t_2, \ldots, t_r} \frac{\delta(p^{t_1}, p^{t_2}, \ldots, p^{t_r})}{p^{t_1 s_1 + t_2 s_2 + \ldots + t_r s_r}} =$$

$$\frac{2C(q)}{\pi} \int_0^{\pi} \sin^2\theta \prod_{j=1}^{r} \frac{e^{i\theta}\left(1 - \frac{e^{i\theta}}{p^{s_j}}\right)^{-1} - e^{-i\theta}\left(1 - \frac{e^{-i\theta}}{p^{s_j}}\right)^{-1}}{e^{i\theta} - e^{-i\theta}} d\theta. \tag{3.17}$$

**Lemma 3.2.4.** *(Lemma 2.5.2 of [7]) Suppose $F$ is a symmetric function of $r$ variables, regular near $(0,0,\dots,0)$, and $f(s)$ has a simple pole of residue 1 at $s=0$ and is otherwise analytic in a neighbourhood of $s=0$, and let*

$$K(a_1,\dots,a_r) = F(a_1,\dots,a_r)\prod_{1\leqslant i\leqslant j\leqslant r} f(a_i+a_j) \tag{3.18}$$

*or*

$$K(a_1,\dots,a_r) = F(a_1,\dots,a_r)\prod_{1\leqslant i< j\leqslant r} f(a_i+a_j). \tag{3.19}$$

*If $\alpha_i+\alpha_j$ are contained in the region of analyticity of $f(s)$ then*

$$\sum_{\epsilon_j=\pm1} K(\epsilon_1\alpha_1,\dots,\epsilon_r\alpha_r) =$$

$$\frac{(-1)^{r(r-1)/2}}{(2\pi i)^r}\frac{2^r}{r!}\oint\dots\oint K(z_1,\dots,z_r)\frac{\Delta(z_1^2,\dots,z_r^2)^2\prod_{j=1}^r z_j}{\prod_{i=1}^r\prod_{j=1}^r(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\dots dz_r, \tag{3.20}$$

*and*

$$\sum_{\epsilon_j=\pm1}(\prod_{j=1}^r\epsilon_j)K(\epsilon_1\alpha_1,\dots,\epsilon_r\alpha_r) =$$

$$\frac{(-1)^{r(r-1)/2}}{(2\pi i)^r}\frac{2^r}{r!}\oint\dots\oint K(z_1,\dots,z_r)\frac{\Delta(z_1^2,\dots,z_r^2)^2\prod_{j=1}^r \alpha_j}{\prod_{i=1}^r\prod_{j=1}^r(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\dots dz_r, \tag{3.21}$$

*where the path of integration encloses $\pm\alpha_j$'s.*

## 3.3 Conjectures

### *3.3.1 The general case*

We follow step by step the recipe given in [7] (section 4.1).

1. Consider a product of $r$ shifted $L$-functions

$$L(s,\alpha_1,\alpha_2,\dots,\alpha_r) := L(s+\alpha_1,f)L(s+\alpha_2,f)\dots L(s+\alpha_r,f). \tag{3.22}$$

2. The precise version of approximate functional equation can be found in [16] (Theorem 5.3). For our purposes the following form is sufficient

$$L(s,f) = \sum \frac{\lambda_f(n)}{n^s} + \epsilon_f X_f(s) \sum \frac{\overline{\lambda_f}(n)}{n^{1-s}} + remainder. \tag{3.23}$$

Since $f \in H_k^*(q)$, coefficients $\lambda_f(n) \in \mathbb{R}$ and $\overline{\lambda_f}(n) = \lambda_f(n)$.

3. Each $L$-function can be replaced with the two order terms of (3.23), ignoring remainder. Multiplying out the resulting expression, one obtains $2^r$ terms of the form

$$(\epsilon_f)^{r-l}(\prod_{j=l+1}^{r} X_f(s+\alpha_j)) \sum_{n_1,n_2,\ldots,n_r} \frac{\lambda_f(n_1)\lambda_f(n_2)\ldots\lambda_f(n_r)}{n_1^{s+\alpha_1}\ldots n_l^{s+\alpha_l} n_{l+1}^{1-s-\alpha_{l+1}}\ldots n_r^{1-s-\alpha_r}} \tag{3.24}$$

for $l = 0, 1, \ldots, r$. Note that

$$X_f(s) = X_f(1-s)^{-1}. \tag{3.25}$$

Also, if we set $s = 1/2$, expression (3.24) is equivalent to

$$(\epsilon_f)^{r-l}(\prod_{j=l+1}^{r} X_f(s-\alpha_j)^{-1}) \sum_{n_1,n_2,\ldots,n_r} \frac{\lambda_f(n_1)\lambda_f(n_2)\ldots\lambda_f(n_r)}{n_1^{s+\alpha_1}\ldots n_l^{s+\alpha_l} n_{l+1}^{s-\alpha_{l+1}}\ldots n_r^{s-\alpha_r}} \tag{3.26}$$

and

$$\prod_{j=l+1}^{r} X_f(s-\alpha_j)^{-1} = \prod_{j=1}^{r} X_f(s-\alpha_j)^{-1/2} \prod_{j=1}^{l} X_f(s+\alpha_j)^{-1/2} \prod_{j=l+1}^{r} X_f(s-\alpha_j)^{-1/2}. \tag{3.27}$$

4. Next, we replace each product of $\epsilon_f$ by its expected value when averaged over the family. For orthogonal family, $\epsilon_f$ is randomly $\pm 1$. Thus, $\langle \epsilon_f \rangle = 0$ unless $r - l$ is even. This gives $2^{r-1}$ terms in the final expression.

5. Finally, the product $\lambda_f(n_1)\lambda_f(n_2)\ldots\lambda_f(n_r)$ is replaced by its expected value

$$\delta(n_1, n_2, \ldots, n_r) = \langle \lambda_f(n_1)\lambda_f(n_2)\ldots\lambda_f(n_r) \rangle.$$

When $n_1, n_2, \ldots, n_r$ are integral, the value of $\delta(n_1, n_2, \ldots, n_r)$ is multiplicative:

$$\delta(n_1 m_1, n_2 m_2, \ldots, n_r m_r) = \delta(n_1, n_2, \ldots, n_r)\delta(m_1, m_2, \ldots, m_r) \tag{3.28}$$

if $(n_1 n_2 \ldots n_r, m_1 m_2 \ldots m_r) = 1$.

Thus, in expression (3.26)

$$\sum_{n_1, n_2, \ldots, n_r} \frac{\lambda_f(n_1)\lambda_f(n_2)\ldots\lambda_f(n_r)}{n_1^{s+\alpha_1}\ldots n_l^{s+\alpha_l} n_{l+1}^{s-\alpha_{l+1}}\ldots n_r^{s-\alpha_r}}$$

can be replaced by

$$\sum_{n_1, n_2, \ldots, n_r} \frac{\delta(n_1, n_2, \ldots, n_r)}{n_1^{s+\alpha_1}\ldots n_l^{s+\alpha_l} n_{l+1}^{s-\alpha_{l+1}}\ldots n_r^{s+\alpha_r}} = R(s, \alpha_1, \ldots, \alpha_l, -\alpha_{l+1}, \ldots, -\alpha_r), \tag{3.29}$$

where

$$R(s, \alpha_1, \alpha_r, \ldots, \alpha_r) := \prod_p \sum_{t_1, t_2, \ldots, t_r} \frac{\delta(p^{t_1}, p^{t_2}, \ldots, p^{t_r})}{p^{t_1(s+\alpha_1)+t_2(s+\alpha_2)+\ldots+t_r(s+\alpha_r)}}. \tag{3.30}$$

6. Lemma 3.2.2 allows us to compute $\delta(p^{t_1}, p^{t_2}, \ldots, p^{t_r})$.

If $p|q$, then $\delta(p^{t_1}, p^{t_2}, \ldots, p^{t_r}) = 0$. Assume that $p \nmid q$. For any $j, i = 1, 2, \ldots, r$, we have

$$\delta(1, 1, \ldots, 1) = C(q),$$

$$\delta(1, \ldots, p^{t_j}, \ldots 1) = 0 \text{ if } t_j = 1,$$

$$\delta(1, \ldots, p^{t_j}, \ldots, p^{t_i}, \ldots 1) = C(q) \text{ if } i \neq j \text{ and } t_i = t_j = 1,$$

$$\delta(1, \ldots, p^{t_j}, \ldots 1) = 0 \text{ if } t_j = 2.$$

Therefore,

$$R(s, \alpha_1, \alpha_r, \ldots, \alpha_r) = C(q) \prod_{p \nmid q} \left(1 + \sum_{1 \leqslant i < j \leqslant q} \frac{1}{p^{2s+\alpha_i+\alpha_j}} + O(p^{-3s+\epsilon})\right) =$$

$$C(q) \prod_{p \nmid q} \left[\prod_{1 \leqslant i < j \leqslant r} \left(1 + \frac{1}{p^{2s+\alpha_i+\alpha_j}}\right) \times (1 + O(p^{-3s+\epsilon}))\right].$$

The product $\prod_{p\nmid q}(1+O(p^{-3s+\epsilon}))$ is regular for $\Re s > 1/3$. And

$$\prod_{p\nmid q}\left(1+\frac{1}{p^{2s+\alpha_i+\alpha_j}}\right)$$

has a simple pole at $s = 1/2 - 1/2(\alpha_i+\alpha_j)$.

To apply Lemma 3.2.4, one needs to separate a polar part of $R(s,\alpha_1,\alpha_r,\ldots,\alpha_r)$. This gives

$$R(s,\alpha_1,\alpha_r,\ldots,\alpha_r) = \prod_{1\leqslant i<j\leqslant r}\zeta_q(2s+\alpha_i+\alpha_j)$$

$$\times\left[\prod_{p\nmid q}\prod_{1\leqslant i<j\leqslant r}\left(1-\frac{1}{p^{2s+\alpha_i+\alpha_j}}\right)\sum_{t_1,t_2,\ldots,t_r}\frac{\delta(p^{t_1},p^{t_2},\ldots,p^{t_r})}{p^{t_1(s+\alpha_1)+t_2(s+\alpha_2)+\ldots+t_r(s+\alpha_r)}}\right].$$

7. According to corollary 3.2.3

$$\sum_{t_1,t_2,\ldots,t_r}\frac{\delta(p^{t_1},p^{t_2},\ldots,p^{t_r})}{p^{t_1(s+\alpha_1)+t_2(s+\alpha_2)+\ldots+t_r(s+\alpha_r)}} =$$

$$\frac{2C(q)}{\pi}\int_0^\pi \sin^2\theta\prod_{j=1}^r\frac{e^{i\theta}\left(1-\frac{e^{i\theta}}{p^{s+\alpha_j}}\right)^{-1}-e^{-i\theta}\left(1-\frac{e^{-i\theta}}{p^{s+\alpha_j}}\right)^{-1}}{e^{i\theta}-e^{-i\theta}}d\theta. \tag{3.31}$$

8. Summing all $2^{r-1}$ terms, we have

$$M(s,\alpha_1,\alpha_2,\ldots,\alpha_r) = C(q)\prod_{j=1}^r X_f(s-\alpha_j)^{-1/2}$$

$$\times\sum_{\substack{\epsilon_j=\pm1\\ \prod_{j=1}^r\epsilon_j=1}}\prod_{j=1}^r X_f(s+\epsilon_j\alpha_j)^{-1/2}\prod_{1\leqslant i<j\leqslant r}\zeta_q(2s+\epsilon_i\alpha_i+\epsilon_j\alpha_j)\prod_{p\nmid q}\prod_{1\leqslant i<j\leqslant r}\left(1-\frac{1}{p^{2s+\epsilon_i\alpha_i+\epsilon_j\alpha_j}}\right)$$

$$\times\prod_{p\nmid q}\frac{2}{\pi}\int_0^\pi\sin^2\theta\prod_{j=1}^r\frac{e^{i\theta}\left(1-\frac{e^{i\theta}}{p^{s+\epsilon_j\alpha_j}}\right)^{-1}-e^{-i\theta}\left(1-\frac{e^{-i\theta}}{p^{s+\epsilon_j\alpha_j}}\right)^{-1}}{e^{i\theta}-e^{-i\theta}}d\theta.$$

And our conjecture is the following:

$$\sum_{f\in H_k^*(q)}^{h} L(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r)=M(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r)+error.$$

9. Now we consider odd and even cases separately. Approximately one half of $L$-functions will have an even symmetry type. Another half will be odd, vanishing at the critical point $s=1/2$. Let

$$A_r(z_1,z_2,\ldots,z_r):=\prod_{p\nmid q}\prod_{1\leqslant i<j\leqslant r}\left(1-\frac{1}{p^{1+z_i+z_j}}\right)$$

$$\times\frac{2}{\pi}\int_0^\pi \sin^2\theta\prod_{j=1}^{r}\frac{e^{i\theta}\left(1-\frac{e^{i\theta}}{p^{1/2+z_j}}\right)^{-1}-e^{-i\theta}\left(1-\frac{e^{-i\theta}}{p^{1/2+z_j}}\right)^{-1}}{e^{i\theta}-e^{-i\theta}}d\theta$$

and

$$T(z_1,z_2,\ldots,z_r):=A_r(z_1,z_2,\ldots,z_r)\prod_{1\leqslant i<j\leqslant r}\zeta_q(1+z_i+z_j). \tag{3.32}$$

Applying Lemma 3.2.4, we have

$$\sum_{\epsilon_j=\pm1}\prod_{j=1}^{r}X_f(1/2+\epsilon_j\alpha_j)^{-1/2}T(\epsilon_1\alpha_1,\ldots,\epsilon_r\alpha_r)=\frac{(-1)^{r(r-1)/2}}{(2\pi i)^r}\frac{2^r}{r!}$$
$$\times\oint\ldots\oint\prod_{j=1}^{r}X_f(1/2+z_j)^{-1/2}T(z_1,\ldots,z_r)\frac{\Delta(z_1^2,\ldots,z_r^2)^2\prod_{j=1}^r z_j}{\prod_{i=1}^r\prod_{j=1}^r(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\ldots dz_r \tag{3.33}$$

and

$$\sum_{\epsilon_j=\pm1}\prod_{j=1}^{r}\epsilon_j X_f(1/2+\epsilon_j\alpha_j)^{-1/2}T(\epsilon_1\alpha_1,\ldots,\epsilon_r\alpha_r)=\frac{(-1)^{r(r-1)/2}}{(2\pi i)^r}\frac{2^r}{r!}$$
$$\times\oint\ldots\oint\prod_{j=1}^{r}X_f(1/2+z_j)^{-1/2}T(z_1,\ldots,z_r)\frac{\Delta(z_1^2,\ldots,z_r^2)^2\prod_{j=1}^r \alpha_j}{\prod_{i=1}^r\prod_{j=1}^r(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\ldots dz_r. \tag{3.34}$$

10. Finally, we obtain conjectures for moments of $L$-functions associated to even and odd primitive forms:

$$\sum_{\substack{f\in H_k^*(q)\\ f \text{ is even}}}^{h} L(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) = 1/2M_1(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) + error,$$

where

$$M_1(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) = \frac{(-1)^{r(r-1)/2}}{(2\pi i)^r}\frac{2^r}{r!}C(q)\prod_{j=1}^{r} X_f(1/2-\alpha_j)^{-1/2}$$
$$\times \oint\cdots\oint \prod_{j=1}^{r} X_f(1/2+z_j)^{-1/2} T(z_1,\ldots,z_r)\frac{\Delta(z_1^2,\ldots,z_r^2)^2\prod_{j=1}^{r} z_j}{\prod_{i=1}^{r}\prod_{j=1}^{r}(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\ldots dz_r;$$

$$\sum_{\substack{f\in H_k^*(q)\\ f \text{ is odd}}}^{h} L(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) = 1/2M_2(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) + error,$$

where

$$M_2(1/2,\alpha_1,\alpha_2,\ldots,\alpha_r) = \frac{(-1)^{r(r-1)/2}}{(2\pi i)^r}\frac{2^r}{r!}C(q)\prod_{j=1}^{r} X_f(1/2-\alpha_j)^{-1/2}$$
$$\times \oint\cdots\oint \prod_{j=1}^{r} X_f(1/2+z_j)^{-1/2} T(z_1,\ldots,z_r)\frac{\Delta(z_1^2,\ldots,z_r^2)^2\prod_{j=1}^{r} \alpha_j}{\prod_{i=1}^{r}\prod_{j=1}^{r}(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\ldots dz_r.$$

### *3.3.2 The fourth moment at the critical point*

By letting shifts tend to zero, we find a more explicit conjecture for the fourth moment.

**Conjecture 3.3.1.** *Let $q=\rho^\nu$ with $\nu \geqslant 2$. Then $M(1/2,0,0,0,0)$ is a polynomial of degree 6 in $\log\hat{q}$ and leading coefficient is*

$$C(q)\left(1-\frac{1}{\rho}\right)^6\frac{a_4b_4}{2},$$

*where*

$$a_4 = \prod_{p \nmid q} \left(1 - \frac{1}{p}\right)^6 \times \frac{2}{\pi} \int_0^{\pi} \frac{p^4 \sin^2 \theta d\theta}{(p + 1 - 2\sqrt{p} \cos\theta)^4} = \frac{1}{\zeta(2)} \frac{\rho^2}{\rho^2 - 1},$$

$$b_4 = \frac{2^4}{45}.$$

**Remark 3.3.2.** *The coefficient $b_4$ is consistent with its random matrix analogue. See Theorem 1.5.6 of [7].*

**Remark 3.3.3.** *Recall that $\hat{q} = \frac{\sqrt{q}}{2\pi}$. Therefore, $M(1/2, 0, 0, 0, 0)$ is a polynomial of degree 6 in $\log q$ and leading coefficient is*

$$C(q) \left(1 - \frac{1}{\rho}\right)^6 \frac{a_4 b_4}{2^7}.$$

*For $\nu \geqslant 3$ we recover Theorem 1.0.8.*

Let $\alpha_j = 0$ for $j = 1, 2, \ldots, r$. Reflection formula (A.9) and functional equation (A.4) give

$$\frac{\Gamma(k/2 + z_j)}{\Gamma(k/2 - z_j)} = \frac{(z_j + k/2 - 1)(z_j + k/2 - 2) \ldots (z_j + 1)}{(-z_j + k/2 - 1)(-z_j + k/2 - 2) \ldots (-z_j + 1)} \frac{z_j \Gamma(z_j)^2 \sin \pi z_j}{\pi}.$$

For $|z_j - 1| < 1$, relation (A.8) implies

$$\frac{\Gamma(k/2 + z_j)}{\Gamma(k/2 - z_j)} = 1 + O(z_j^3).$$

Thus,

$$X_f(1/2 + z_j)^{-1/2} = e^{z_j \log \hat{q}} \left(\frac{\Gamma(k/2 + z_j)}{\Gamma(k/2 - z_j)}\right)^{1/2} = e^{z_j \log \hat{q}}(1 + O(z_j^3)).$$

Replacing $X_f(1/2 + z_j)^{-1/2}$ by $e^{z_j \log \hat{q}}$, we have

$$M(1/2, 0, 0, \ldots, 0) = \frac{(-1)^{r(r-1)/2}}{(2\pi i)^r} \frac{2^{r-1}}{r!} C(q)$$

$$\times \oint \dots \oint e^{\sum_{j=1}^{r} z_j \log \hat{q}} T(z_1, \dots, z_r) \frac{\Delta(z_1^2, \dots, z_r^2)^2}{\prod_{j=1}^{r} z_j^{2r-1}} dz_1 \dots dz_r.$$

Let $x := \log \hat{q}$. We change variables

$$z_j \to z_j/x$$

so that

$$M(1/2, 0, 0, \dots, 0) = \frac{(-1)^{r(r-1)/2}}{(2\pi i)^r} \frac{2^{r-1}}{r!} C(q)$$

$$\times \oint \dots \oint e^{\sum_{j=1}^{r} z_j} A_r(z_1/x, \dots, z_r/x) \prod_{1 \leqslant i < j \leqslant r} \zeta_q \left(1 + \frac{z_i + z_j}{x}\right) \frac{\Delta(z_1^2, \dots, z_r^2)^2}{\prod_{j=1}^{r} z_j^{2r-1}} dz_1 \dots dz_r.$$

Let $r = 4$, then

$$M_4(x) := M(1/2, 0, 0, 0, 0) = \frac{1}{(2\pi i)^4} \frac{2^3}{4!} C(q)$$

$$\times \oint \dots \oint e^{\sum_{j=1}^{4} z_j} A_4(z_1/x, \dots, z_4/x) \prod_{1 \leqslant i < j \leqslant 4} \zeta_q \left(1 + \frac{z_i + z_j}{x}\right) \frac{\Delta(z_1^2, \dots, z_4^2)^2}{\prod_{j=1}^{4} z_j^{7}} dz_1 \dots dz_4.$$

The $\zeta$ function has a simple pole at 1. Thus,

$$M_4(x) = \frac{A_4(0,0,0,0)}{(2\pi i)^4} \frac{2^3}{4!} C(q) \left(1 - \frac{1}{\rho}\right)^6 x^6 (1 + O(x^{-1}))$$

$$\times \oint \dots \oint e^{\sum_{j=1}^{4} z_j} \frac{\Delta(z_1^2, \dots, z_4^2)^2}{\prod_{1 \leqslant i < j \leqslant 4} (z_i + z_j) \prod_{j=1}^{4} z_j^7} dz_1 \dots dz_4$$

$$= \frac{A_4(0,0,0,0)}{(2\pi i)^4} \frac{2^3}{4!} C(q) \left(1 - \frac{1}{\rho}\right)^6 x^6 (1 + O(x^{-1}))$$

$$\times \oint \dots \oint e^{\sum_{j=1}^{4} z_j} \frac{\Delta(z_1^2, \dots, z_4^2) \Delta(z_1, \dots, z_4)}{\prod_{j=1}^{4} z_j^7} dz_1 \dots dz_4.$$

Let

$$a_4 := A_4(0,0,0,0) = \prod_{p \nmid q} \left(1 - \frac{1}{p}\right)^6 \times \frac{2}{\pi} \int_0^{\pi} \frac{p^4 \sin \theta^2 d\theta}{(p + 1 - 2\sqrt{p} \cos \theta)^4}$$

$$= \prod_{p \nmid q} \left(1 - \frac{1}{p}\right)^6 \frac{(p+1)p^4}{(p-1)^5} = \frac{1}{\zeta(2)} \frac{\rho^2}{\rho^2 - 1}.$$

Then the leading coefficient corresponding to $(\log \hat{q})^6$ equals

$$c_4 = 1/2C(q) \left(1 - \frac{1}{\rho}\right)^6 a_4 b_4$$

with

$$b_4 := \lim_{x\to\infty} \frac{M_4(x)}{1/2C(q)\left(1-\frac{1}{\rho}\right)^6 a_4 x^6} = \frac{1}{(2\pi i)^4} \frac{2^4}{4!}$$

$$\times \oint \dots \oint e^{\sum_{j=1}^4 z_j} \sum_S \operatorname{sgn}(S) z_1^{2S_0} z_2^{2S_1} z_3^{2S_2} z_4^{2S_3} \sum_T \operatorname{sgn}(T) z_1^{T_0} z_2^{T_1} z_3^{T_2} z_4^{T_3} \prod_{j=1}^4 z_j^{-7} dz_1 \dots dz_4$$

$$= \frac{2^4}{(2\pi i)^4} \times \oint \dots \oint e^{\sum_{j=1}^4 z_j} \sum_S \operatorname{sgn}(S) z_1^{-(7-2S_0)} z_2^{-(6-2S_1)} z_3^{-(5-2S_2)} z_4^{-(4-2S_3)} dz_1 \dots dz_4.$$

Finally,

$$b_4 = 2^4 \sum_S \operatorname{sgn}(S) \frac{1}{\Gamma(7-2S_0)\Gamma(6-2S_1)\Gamma(5-2S_2)\Gamma(4-2S_3)}$$

$$= 2^4 \begin{vmatrix} \frac{1}{\Gamma(7)} & \frac{1}{\Gamma(6)} & \frac{1}{\Gamma(5)} & \frac{1}{\Gamma(4)} \\ \frac{1}{\Gamma(5)} & \frac{1}{\Gamma(4)} & \frac{1}{\Gamma(3)} & \frac{1}{\Gamma(2)} \\ \frac{1}{\Gamma(3)} & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} & \frac{1}{\Gamma(0)} \\ \frac{1}{\Gamma(1)} & \frac{1}{\Gamma(0)} & \frac{1}{\Gamma(-1)} & \frac{1}{\Gamma(-2)} \end{vmatrix} = \frac{2^4}{45}.$$

# 4

# The fourth moment: diagonal and off-diagonal terms

Let $q = \rho^\nu$, where $\rho$ is a fixed prime number and $\nu \to \infty$. Our goal is to verify heuristic predictions of random matrix theory for the fourth moment of $L$-functions associated to primitive forms of level $q$ and weight $k \geqslant 2$. Combinatorial structure of mean values is more clearly revealed if each $L$-function in a product is slightly shifted from the critical line $\Re s = 1/2$. We consider

$$M_4 = M_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{f \in H_k^*(q)}^{h} L(1/2+\alpha_1, f)L(1/2+\alpha_2, f)L(1/2+\alpha_3, f)L(1/2+\alpha_4, f),$$

where $\alpha_1 := t_1 + ir_1$, $\alpha_2 := t_1 - ir_1$, $\alpha_3 := t_2 + ir_2$, $\alpha_4 := t_2 - ir_2$, $|t_1| < 1/2$, $|t_2| < 1/2$ and $t_1, t_2, r_1, r_2 \in \mathbb{R}$.

In this chapter, we decompose the main term of $M_4$ into diagonal $M^D$, off-diagonal $M^{OD}$ and off-off-diagonal $M^{OD}$ parts. Further, we prove an asymptotic formula for the diagonal and off-diagonal terms.

**Theorem 4.0.4.** *For all $\epsilon > 0$, up to an error term*

$$O_{\epsilon,\rho,t_1,t_2}(P(r_1)P(r_2))q^{|t_1|-t_1+|t_2|-t_2+\epsilon}(q^{-\frac{2k-3}{12}} + q^{-1/4})),$$

*we have*

$$M^D + M^{OD} = \frac{\phi(q)}{q} \sum_{\epsilon_1,\epsilon_2=\pm 1} \hat{q}^{-2t_1-2t_2+2\epsilon_1 t_1+2\epsilon_2 t_2}$$

$$\times \zeta_q(1+2\epsilon_1 t_1)\zeta_q(1+2\epsilon_2 t_2)\frac{\prod \zeta_q(1+\epsilon_1 t_1+\epsilon_2 t_2 \pm ir_1 \pm ir_2)}{\zeta_q(2+2\epsilon_1 t_1+2\epsilon_2 t_2)}$$

$$\times \frac{\Gamma(\epsilon_1 t_1+ir_1+k/2)\Gamma(\epsilon_1 t_1-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\frac{\Gamma(\epsilon_2 t_2+ir_2+k/2)\Gamma(\epsilon_2 t_2-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}. \quad (4.1)$$

**Remark 4.0.5.** *The biggest error term appears in Lemmas 4.4.4, 5.3.1.*

By letting shifts tend to zero, we obtain an asympotic formula at the critical point.

**Theorem 4.0.6.** *For all $\epsilon > 0$,*

$$M^D + M^{OD} = Q(\log q) + O_{\epsilon,\rho}(q^{\epsilon}(q^{-\frac{2k-3}{12}} + q^{-1/4})), \quad (4.2)$$

*where $Q$ is a polynomial of degree 6 with leading coefficient*

$$(\frac{\phi(q)}{q})^7 \frac{\rho^2}{\rho^2-1}\frac{1}{60\pi^2}. \quad (4.3)$$

## 4.1 Approximate functional equation

Let

$$\tau_v(n) = |n|^{v-1/2}\sigma_{1-2v}(n) = |n|^{v-1/2} \sum_{d|n,d>0} d^{1-2v}. \quad (4.4)$$

If $v = 1/2$, then $\tau_v(n)$ reduces to the divisor function $\tau(n)$. Furthemore, $\tau_v(n)$ satisfies the property of multiplicity (see [27], page 74)

$$\tau_v(n)\tau_v(m) = \sum_{d|(n,m)} \tau_v\left(\frac{nm}{d^2}\right). \quad (4.5)$$

**Lemma 4.1.1.** *(Ramanujan's identity, [39], page 8)*

*Let $\Re s > 1 + |\Re v - 1/2| + |\Re \mu - 1/2|$. Then*

$$\zeta(2s)\sum_{n\geqslant 1}\frac{\tau_v(n)\tau_\mu(n)}{n^s} = \zeta(s+v-\mu)\zeta(s-v+\mu)\zeta(s+v+\mu-1)\zeta(s-v-\mu+1). \quad (4.6)$$

*If $v = \mu = 1/2$, this reduces to*

$$\sum_{n\geqslant 1} \frac{\tau(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)}. \tag{4.7}$$

**Lemma 4.1.2.** *For $\Re s > 1/2$, we have*

$$L(1/2+s+ir,f)L(1/2+s-ir,f) = \zeta_q(1+2s)\sum_{n\geqslant 1}\frac{\lambda_f(n)}{n^{1/2+s}}\tau_{1/2+ir}(n), \tag{4.8}$$

*Proof.* Consider

$$L(1/2+s+ir,f)L(1/2+s-ir,f) = \sum_{\substack{a,b,d\geqslant 1\\(d,q)=1}} \frac{\lambda_f(ab/d^2)}{a^{1/2+s+ir}b^{1/2+s-ir}d^{1+2s}}$$

$$= \zeta_q(1+2s)\sum_{n\geqslant 1}\frac{\lambda_f(n)}{n^{1/2+s}}\tau_{1/2+ir}(n).$$

□

Let $G(s)$ be an even polynomial vanishing at all poles of $\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)$ in the range $\Re s \geqslant -L$ for some large constant $L > 0$. We define

$$W_{t,r}(y) := \frac{1}{2\pi i}\int_{(3)} \frac{G(s)}{G(t)}\zeta_q(1+2s)\frac{\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)}y^{-s}\frac{2s ds}{s^2-t^2}. \tag{4.9}$$

**Lemma 4.1.3.** *Suppose $y > 0$, $|t| < 1/2$. For any $C > |t|$*

$$W_{t,r}(y) = O_{C,t}(P(r)y^{-C}) \text{ as } y \to \infty, \tag{4.10}$$

$$\begin{aligned} W_{t,r}(y) &= \zeta_q(1+2t)y^{-t} \\ &\quad + \zeta_q(1-2t)y^t\frac{\Gamma(-t+ir+k/2)\Gamma(-t-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} + O_{C,t}(P(r)y^C) \text{ as } y \to 0. \end{aligned} \tag{4.11}$$

**Remark 4.1.4.** *Notation $P(r)$ means polynomial dependence on $r$.*

*Proof.* Asymptotic expansion for the ratio of gamma functions (A.7) gives

$$\frac{\Gamma(C+ir+k/2)\Gamma(C-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} = (|r|)^{2(C-t)}\left(1+O(1/|r|)\right).$$

First, without crossing any pole, we can shift the contour of integration to $\Re s = C$ with $C > |t|$. This implies (4.10).

Second, we move the contour of integration to $\Re s = -C$, meeting two simple poles at $s = \pm t$. Therefore, as $y \to 0$, we have

$$W_{t,r}(y) = \zeta_q(1+2t)y^{-t} + \zeta_q(1-2t)y^t \frac{\Gamma(-t+ir+k/2)\Gamma(-t-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} + O_{C,t}(P(r)y^C).$$

$\square$

**Lemma 4.1.5.** *For $t, r \in \mathbb{R}$ with $|t| < 1/2$, we have*

$$L(1/2+t+ir,f)L(1/2+t-ir,f) = (\hat{q})^{-2t}\sum_{n\geqslant 1}\tau_{1/2+ir}(n)\frac{\lambda_f(n)}{\sqrt{n}}W_{t,r}\left(\frac{n}{\hat{q}^2}\right). \tag{4.12}$$

*Proof.* Consider

$$I_t := \frac{1}{2\pi i}\int_{(3)}\Lambda(1/2+s+ir,f)\Lambda(1/2+s-ir,f)\frac{G(s)}{s-t}ds.$$

Moving the contour of integration to $\Re s = -3$, we pick up a simple pole at $s = t$. The functional equation (2.23) imples that

$$\begin{aligned} I_t + \epsilon_f^2 I_{-t} &= Res_{s=t}\left(\Lambda(1/2+s+ir,f)\Lambda(1/2+s-ir,f)\frac{G(s)}{s-t}\right) \\ &= G(t)\Lambda(1/2+t+ir,f)\Lambda(1/2+t-ir,f). \end{aligned}$$

Therefore, by Lemma 4.1.2

$$L(1/2+t+ir,f)L(1/2+t-ir,f) = (\hat{q})^{-2t}\sum_{n\geqslant 1}\tau_{1/2+ir}(n)\frac{\lambda_f(n)}{\sqrt{n}}$$

$$\times \frac{1}{2\pi i}\int_{(3)}\frac{G(s)}{G(t)}\zeta_q(1+2s)\frac{\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)}\left(\frac{n}{\hat{q}^2}\right)^{-s}\frac{2sds}{s^2-t^2}.$$

$\square$

**Proposition 4.1.6.** *The fourth moment can be written as follows*

$$M_4 = \hat{q}^{-2t_1-2t_2} \sum_{m,n \geqslant 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) \frac{1}{\sqrt{mn}} W_{t_1,r_1}\left(\frac{m}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{n}{\hat{q}^2}\right) \Delta_q^*(m,n). \quad (4.13)$$

*Proof.* By Lemma 4.1.5

$$L(1/2+t_1+ir_1,f)L(1/2+t_1-ir_1,f)L(1/2+t_2+ir_2,f)L(1/2+t_2-ir_2,f) =$$

$$\hat{q}^{-2t_1-2t_2} \sum_{m,n \geqslant 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) \frac{\lambda_f(m)\lambda_f(n)}{\sqrt{mn}} W_{t_1,r_1}\left(\frac{m}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{n}{\hat{q}^2}\right).$$

Summing over $f \in H_k^*(q)$, one has

$$M_4 = \hat{q}^{-2t_1-2t_2} \sum_{m,n \geqslant 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) \frac{1}{\sqrt{mn}} W_{t_1,r_1}\left(\frac{m}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{n}{\hat{q}^2}\right) \Delta_q^*(m,n).$$

□

## 4.2 Applying the Petersson trace formula

Here we apply Theorem 2.4.2 for $\nu \geqslant 3$. The case $\nu = 2$ can be treated similarly, but it doesn't seem to be of particular interest since the final goal is $\nu = \infty$. Let

$$T(c) := c \sum_{\substack{m,n \\ (q,mn)=1}} \frac{\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)}{\sqrt{nm}} W_{t_1,r_1}(\frac{m}{\hat{q}^2}) W_{t_2,r_2}(\frac{n}{\hat{q}^2}) S(m,n,c) J_{k-1}(\frac{4\pi\sqrt{mn}}{c}). \quad (4.14)$$

Using the trace formula (2.30), the fourth moment (4.13) can be written as a sum of diagonal and non-diagonal parts.

**Proposition 4.2.1.**

$$M_4 = M^D + M_1^{ND} + M_2^{ND}, \quad (4.15)$$

*where*

$$M^D = \frac{\phi(q)}{q} \hat{q}^{-2t_1-2t_2} \sum_{(q,n)=1} \frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n} W_{t_1,r_1}(\frac{n}{\hat{q}^2}) W_{t_2,r_2}(\frac{n}{\hat{q}^2}), \quad (4.16)$$

$$M_1^{ND} = 2\pi i^{-k}\hat{q}^{-2t_1-2t_2}\sum_{q|c}\frac{1}{c^2}T(c), \tag{4.17}$$

*and*

$$M_2^{ND} = -\frac{2\pi i^{-k}}{\rho}\hat{q}^{-2t_1-2t_2}\sum_{\frac{q}{\rho}|c}\frac{1}{c^2}T(c). \tag{4.18}$$

We study the given sums using techniques developed by Duke, Friedlander, Iwaniec in [11] and Kowalski, Michel, Vanderkam in [24]. Accordingly, we split the non-diagonal terms into off-diagonal and off-off-diagonal parts. As expected, the main technical difficulty is caused by the off-off-diagonal term. In [24] (when $q$ is prime), the off-off-diagonal term was further separated into two parts. The first one was evaluated using Theorem 5.1.1 of [11], which is based on $\delta$-symbol method. And the second part was shown to be an error term in section 4.4. This is not the case when $q$ is a prime power. Accordingly, we apply $\delta$-symbol method to the whole off-off-diagonal term.

## 4.3 Estimation of the diagonal term

**Lemma 4.3.1.** *One has*

$$M^D \ll_{\epsilon,t_1,t_2} P(r_1)P(r_2)\frac{\phi(q)}{q}q^{\epsilon-t_1-t_2} \text{ for any } \epsilon > 0. \tag{4.19}$$

*Proof.* Consider

$$M^D = \frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2}\sum_{\substack{n\geqslant 1\\(q,n)=1}}\frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n}W_{t_1,r_1}(\frac{n}{\hat{q}^2})W_{t_2,r_2}(\frac{n}{\hat{q}^2}).$$

Note that $\tau_{1/2+ir}(n) \ll n^\epsilon$ for all $\epsilon > 0$. The sum over $n$ can be decomposed into two cases

$$\sum_{\substack{n\geqslant 1\\(q,n)=1}} = \sum_{\substack{1\leqslant n\leqslant \hat{q}^2\\(q,n)=1}} + \sum_{\substack{n>\hat{q}^2\\(q,n)=1}}.$$

According to (4.11),

$$\frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2}\sum_{\substack{1\leqslant n\leqslant \hat{q}^2\\(q,n)=1}}\frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n}W_{t_1,r_1}(\frac{n}{\hat{q}^2})W_{t_2,r_2}(\frac{n}{\hat{q}^2})$$

$$\ll_{\epsilon,t_1,t_2} P(r_1)P(r_2)\frac{\phi(q)}{q}q^{\epsilon-t_1-t_2}.$$

Using (4.10), we estimate the second sum

$$\frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2}\sum_{\substack{n>\hat{q}^2\\(q,n)=1}}\frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n}W_{t_1,r_1}(\frac{n}{\hat{q}^2})W_{t_2,r_2}(\frac{n}{\hat{q}^2})$$

$$\ll_{\epsilon,t_1,t_2} P(r_1)P(r_2)\frac{\phi(q)}{q}q^{\epsilon-t_1-t_2}.$$

It follows that

$$M^D \ll_{\epsilon,t_1,t_2} P(r_1)P(r_2)\frac{\phi(q)}{q}q^{\epsilon-t_1-t_2}.$$

□

**Remark 4.3.2.** *The asymptotics of this term will be evaluated in section 4.8.*

## 4.4 Smooth partition of unity and restriction of summations

In order to simplify computations of the non-diagonal terms (4.17) and (4.18), it is useful to restrict the ranges of summation.

First, following [24], we make a smooth partition of unity

$$\frac{1}{\sqrt{mn}}W_{t_1,r_1}(\frac{m}{\hat{q}^2})W_{t_2,r_2}(\frac{n}{\hat{q}^2}) = \sum_{M,N\geqslant 1}F_{M,N}(m,n),$$

where

$$F_{M,N}(m,n) := \frac{1}{\sqrt{mn}}W_{t_1,r_1}(\frac{m}{\hat{q}^2})W_{t_2,r_2}(\frac{n}{\hat{q}^2})F_M(m)F_N(n). \tag{4.20}$$

We assume that $F_M(m)$ and $F_N(n)$ are compactly supported functions in $[M/2, 3M]$ and $[N/2, 3N]$, such that for any integral $i, j \geqslant 0$

$$x^j F_M^{(j)}(x) \ll_j 1 \text{ and } y^i F_N^{(i)}(y) \ll_i 1. \tag{4.21}$$

The term (4.14) can be written as

$$T(c) = \sum_{M,N \geqslant 1} T_{M,N}(c), \tag{4.22}$$

where

$$T_{M,N}(c) = c \sum_{\substack{m,n \\ (q,mn)=1}} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) S(m,n,c) F_{M,N}(m,n) J_{k-1}(\frac{4\pi\sqrt{mn}}{c}). \tag{4.23}$$

**Lemma 4.4.1.** *For any* $\alpha_1 \geqslant |t_1|$, $\alpha_2 \geqslant |t_2|$

$$x^i y^j \frac{\partial^i}{\partial^i x} \frac{\partial^j}{\partial^j y} F_{M,N}(x,y) \ll_{\alpha_1,\alpha_2,t_1,t_2}$$
$$P(r_1)P(r_2)(MN)^{-1/2} \left(\frac{\hat{q}^2}{x}\right)^{\alpha_1} \left(\frac{\hat{q}^2}{y}\right)^{\alpha_2} \quad \textit{if } M, N \gg q^{1+\epsilon}; \tag{4.24}$$

$$x^i y^j \frac{\partial^i}{\partial^i x} \frac{\partial^j}{\partial^j y} F_{M,N}(x,y) \ll_{\alpha_1,t_1,t_2}$$
$$P(r_1)P(r_2)(MN)^{-1/2} \left(\frac{\hat{q}^2}{x}\right)^{\alpha_1} \left(\frac{\hat{q}^2}{y}\right)^{|t_2|} \quad \textit{if } M \gg q^{1+\epsilon}, N \ll q^{1+\epsilon}; \tag{4.25}$$

$$x^i y^j \frac{\partial^i}{\partial^i x} \frac{\partial^j}{\partial^j y} F_{M,N}(x,y) \ll_{\alpha_2,t_1,t_2}$$
$$P(r_1)P(r_2)(MN)^{-1/2} \left(\frac{\hat{q}^2}{x}\right)^{|t_1|} \left(\frac{\hat{q}^2}{y}\right)^{\alpha_2} \quad \textit{if } M \ll q^{1+\epsilon}, N \gg q^{1+\epsilon}; \tag{4.26}$$

$$x^i y^j \frac{\partial^i}{\partial^i x} \frac{\partial^j}{\partial^j y} F_{M,N}(x,y) \ll_{t_1,t_2}$$
$$P(r_1)P(r_2)(MN)^{-1/2} \left(\frac{\hat{q}^2}{x}\right)^{|t_1|} \left(\frac{\hat{q}^2}{y}\right)^{|t_2|} \quad \textit{if } M, N \ll q^{1+\epsilon}. \tag{4.27}$$

*Proof.* Consider

$$x^i y^j \frac{\partial^i}{\partial^i x}\frac{\partial^j}{\partial^j y}F_{M,N}(x,y) = \left(x^i \frac{\partial^i}{\partial^i x}\frac{1}{\sqrt{x}}W_{t_1,r_1}(\frac{x}{\hat{q}^2})F_M(x)\right)\left(y^j \frac{\partial^j}{\partial^j y}\frac{1}{\sqrt{y}}W_{t_2,r_2}(\frac{y}{\hat{q}^2})F_N(y)\right)$$

Note that

$$x^{i_1}\frac{\partial^{i_1}}{\partial^{i_1} x}\frac{1}{\sqrt{x}} \ll M^{-1/2}.$$

By (4.21),

$$x^{i_2}\frac{\partial^{i_2}}{\partial^{i_2} x}F_M(x) \ll 1.$$

If $M \gg q^{1+\epsilon}$, we use (4.10) to get

$$x^{i_3}\frac{\partial^{i_3}}{\partial^{i_3} x}W_{t_1,r_1}(\frac{x}{\hat{q}^2}) \ll_{\alpha_1,t_1} P(r_1)(\frac{x}{\hat{q}^2})^{-\alpha_1},$$

If $M \ll q^{1+\epsilon}$, we use (4.11) to get

$$x^{i_3}\frac{\partial^{i_3}}{\partial^{i_3} x}W_{t_1,r_1}(\frac{x}{\hat{q}^2}) \ll_{t_1} P(r_1)(\frac{x}{\hat{q}^2})^{-|t_1|}.$$

By Leibniz's rule (D.1),

$$x^i \frac{\partial^i}{\partial^i x}\frac{1}{\sqrt{x}}W_{t_1,r_1}(\frac{x}{\hat{q}^2})F_M(x) \ll_{\alpha_1,t_1} \frac{1}{\sqrt{M}}P(r_1)(\frac{x}{\hat{q}^2})^{-\alpha_1} \text{ if } M \gg q^{1+\epsilon}$$

and

$$x^i \frac{\partial^i}{\partial^i x}\frac{1}{\sqrt{x}}W_{t_1,r_1}(\frac{x}{\hat{q}^2})F_M(x) \ll_{t_1} \frac{1}{\sqrt{M}}P(r_1)(\frac{x}{\hat{q}^2})^{-|t_1|} \text{ if } M \ll q^{1+\epsilon}.$$

□

**Proposition 4.4.2.** *For any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$*

$$\sum_{\max(M,N) \gg q^{1+\epsilon}} \sum_{\frac{q}{\rho^l}|c} \frac{1}{c^2}T_{M,N}(c) \ll_{\epsilon,\rho,A,t_1,t_2} P(r_1)P(r_2)q^{-A}. \tag{4.28}$$

**Corollary 4.4.3.** *The range of summation in (4.22) can be restricted to $M, N \ll q^{1+\epsilon}$.*

*Proof.* Since $\max(M,N) \gg q^{1+\epsilon}$, there are three cases to consider:

- $M \gg q^{1+\epsilon}$, $N \ll q^{1+\epsilon}$;
- $M \ll q^{1+\epsilon}$, $N \gg q^{1+\epsilon}$;
- $M \gg q^{1+\epsilon}$, $N \gg q^{1+\epsilon}$.

We give a proof for the first case. The other two can be treated in the same manner.

$$\sum_{\substack{M \gg q^{1+\epsilon} \\ N \ll q^{1+\epsilon}}} \sum_{\frac{q}{\rho^l}|c} \frac{1}{c^2} T_{M,N}(c) =$$

$$\sum_{\substack{M \gg q^{1+\epsilon} \\ N \ll q^{1+\epsilon}}} \sum_{\frac{q}{\rho^l}|c} \sum_{\substack{m,n \\ (q,mn)=1}} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) \frac{S(m,n,c)}{c} F_{M,N}(m,n) J_{k-1}(\frac{4\pi\sqrt{mn}}{c}).$$

We decompose the sum over $c$ into two cases and apply (C.7) and (2.7):

$$\sum_{\frac{q}{\rho}^l|c} \frac{S(m,n,c)}{c} J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) =$$

$$\sum_{\substack{c<\sqrt{mn} \\ \frac{q}{\rho^l}|c}} \frac{S(m,n,c)}{c} J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) + \sum_{\substack{c \geqslant \sqrt{mn} \\ q/\rho^l|c}} \frac{S(m,n,c)}{c} J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) \ll (mn)^{3/4+\delta}$$

for any $\delta > 0$. Applying Lemma 4.4.1 with $i = j = 0$, one obtains

$$\sum_{\substack{M \gg q^{1+\epsilon} \\ N \ll q^{1+\epsilon}}} \sum_{\frac{q}{\rho^l}|c} \frac{1}{c^2} T_{M,N}(c) \ll_{\alpha_1,\rho,t_1,t_2} P(r_1)P(r_2)(MN)^{1/4+\delta} \left(\frac{q}{M}\right)^{\alpha_1} \left(\frac{q}{N}\right)^{|t_2|}.$$

Taking $\alpha_1$ sufficiently large (for example, $\alpha_1 = 1/2 + 2\delta + \frac{A+2\delta+1/2+|t_2|}{\epsilon}$ ), it follows that for any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$

$$\sum_{\substack{M \gg q^{1+\epsilon} \\ N \ll q^{1+\epsilon}}} \sum_{\frac{q}{\rho^l}|c} \frac{1}{c^2} T_{M,N}(c) \ll_{\epsilon,\rho,A,t_1,t_2} P(r_1)P(r_2)q^{-A}.$$

□

Finally, the range of summation on $c$ can be restricted via the large sieve inequality.

**Lemma 4.4.4.** *Let $l = 0, 1$. Assume that $M, N \ll q^{1+\epsilon}$. For any $C > \sqrt{MN}$ we have*

$$\sum_{\substack{c \geqslant C \\ \frac{q}{\rho^l} | c}} \frac{1}{c^2} T_{M,N}(c) \ll_{\epsilon,\rho,t_1,t_2} P(r_1)P(r_2) \left(\frac{\hat{q}^2}{M}\right)^{|t_1|} \left(\frac{\hat{q}^2}{N}\right)^{|t_2|} q^{\epsilon} \left(\frac{\sqrt{MN}}{C}\right)^{k-3/2}. \tag{4.29}$$

**Remark 4.4.5.** *We choose $C = \min(q^{2/3}M^{1/2}, q^{7/6})$. Thus, the error term is*

$$P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2+\epsilon}q^{-\frac{2k-3}{12}}.$$

*Proof.* We would like to apply Theorem 2.3.2. In order to do so, we make a dyadic partition of the interval $[C, \infty)$ and assume that $c \in [C, 2C]$. By definition,

$$\sum_{\frac{q}{\rho^l}|c} \frac{1}{c^2} T_{M,N}(c) =$$

$$\sum_{\substack{n,m \\ (q,nm)=1}} \sum_{\frac{q}{\rho^l}|c} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)\frac{1}{c}S(m,n,c)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)F_{M,N}(m,n)$$

$$\ll (MN)^{\epsilon}\frac{\rho^l}{q}\sum_{m,n}\sum_{c_1}\frac{1}{c_1}S(m,n,c_1q/\rho^l)J_{k-1}\left(\frac{4\pi\sqrt{mn}\rho^l}{c_1q}\right)F_{M,N}(m,n).$$

Here $m \in [M/2, 3M]$, $n \in [N/2, 3N]$ and $c_1 \in [C_1, 2C_1]$ with $C_1 := C\rho^l/q$. Let

$$X := P(r_1)P(r_2)\left(\frac{\hat{q}^2}{M}\right)^{-|t_1|}\left(\frac{\hat{q}^2}{N}\right)^{-|t_2|}\sqrt{MN}C_1\left(\frac{\sqrt{MN}}{C}\right)^{-k+1}.$$

As a test function we choose

$$g(m,n,c_1) := \frac{X}{c_1}F_{M,N}(m,n)J_{k-1}\left(\frac{4\pi\sqrt{mn}\rho^i}{c_1q}\right).$$

Since $C > \sqrt{MN}$, the following version of (C.7) can be used

$$J_{k-1}^{(j)}(z) \ll z^{k-1-j} \text{ for } z \ll 1.$$

The function $F_{M,N}(m,n)$ can be bounded using (4.27). Then $g(m,n,c_1)$ satisfies condition (2.27), and Theorem 2.3.2 can be applied with $r=1$ and $s=q/\rho^l$. Selberg's bound (2.24) implies that $\theta_{rs}\leqslant 1/2$. Finally,

$$\sum_{\frac{q}{\rho^l}|c}\frac{1}{c^2}T_{M,N}(c)\ll_{\epsilon,\rho}(MN)^\epsilon\frac{\rho^l}{q}\frac{1}{X}\sum_{n,m}\sum_{c_1}S(m,n,c_1q/\rho^l)g(m,n,c_1)\ll_{\epsilon,\rho,t_1,t_2}$$

$$q^\epsilon\frac{\rho^l}{q}\frac{\sqrt{MN}}{X}\left(1+\frac{q\sqrt{C_1}}{\rho^l\sqrt{MN}}\right)^{1/2}\frac{\left(\frac{q}{\rho^l}C_1+\sqrt{MN}+\sqrt{\frac{q}{\rho^l}MC_1}\right)\left(\frac{q}{\rho^l}C_1+\sqrt{MN}+\sqrt{\frac{q}{\rho^l}NC_1}\right)}{\frac{q}{\rho^l}C_1+\sqrt{MN}}.$$

Plugging in the expression for $X$, we have

$$\sum_{\frac{q}{\rho^l}|c}\frac{1}{c^2}T_{M,N}(c)\ll_{\epsilon,\rho,t_1,t_2}q^\epsilon P(r_1)P(r_2)\left(\frac{\hat{q}^2}{M}\right)^{|t_1|}\left(\frac{\hat{q}^2}{N}\right)^{|t_2|}$$

$$\times\frac{1}{C}\left(\frac{\sqrt{MN}}{C}\right)^{k-3/2}\left(\frac{\sqrt{MN}}{C}+\sqrt{\frac{1}{C_1}}\right)^{1/2}\frac{C\rho^l}{q}\sqrt{q/\rho^l+M}\sqrt{q/\rho^l+N}.$$

Conditions $C>\sqrt{MN}$ and $M,N\ll q^{1+\epsilon}$ imply that

$$\left(\frac{\sqrt{MN}}{C}+\sqrt{\frac{1}{C_1}}\right)^{1/2}\ll 1,$$

$$\frac{\rho^l}{q}\sqrt{q/\rho^l+M}\sqrt{q/\rho^l+N}\ll 1.$$

Finally,

$$\sum_{\frac{q}{\rho^l}|c}\frac{1}{c^2}T_{M,N}(c)\ll_{\epsilon,\rho,t_1,t_2}P(r_1)P(r_2)\left(\frac{\hat{q}^2}{M}\right)^{|t_1|}\left(\frac{\hat{q}^2}{N}\right)^{|t_2|}q^\epsilon\left(\frac{\sqrt{MN}}{C}\right)^{k-3/2}.$$

$\square$

## 4.5 Poisson summation formula connected with the Eisenstein-Maass series

In [24], Julila's extension of Voronoi summation formula was used to transform Kloosterman sums into Ramanujan sums.

**Proposition 4.5.1.** *(Jutila, [20], Theorem 1.7) Let $g(x)$ be a smooth, compactly supported function on $\mathbb{R}^+$ and $(c,d)=1$. Then*

$$c\sum_{m\geqslant 1}\tau(m)e(dm/c)g(m)=\int_0^{+\infty}(\log x+2\gamma-2\log c)g(x)dx$$
$$-2\pi\sum_{m\geqslant 1}\tau(m)e(-\bar{d}m/c)\int_0^{+\infty}Y_0(\frac{4\pi\sqrt{mx}}{c})g(x)dx$$
$$+4\sum_{m\geqslant 1}\tau(m)e(\bar{d}m/c)\int_0^{+\infty}K_0(\frac{4\pi\sqrt{mx}}{c})g(x)dx. \quad (4.30)$$

In our case, $\tau(m)$ is replaced by $\tau_{1/2+ir}(m)$.

Consider the Bessel kernels

$$k_0(x,v):=\frac{1}{2\cos\pi v}(J_{2v-1}(x)-J_{1-2v}(x)), \quad (4.31)$$

$$k_1(x,v):=\frac{2}{\pi}\sin\pi v K_{2v-1}(x). \quad (4.32)$$

Let

$$D_v(s,x):=\sum_{n\geqslant 1}\frac{\tau_v(n)}{n^s}e(nx),\ \Re v=1/2. \quad (4.33)$$

The series (4.33) converges absolutely for $\Re s>1$.

**Theorem 4.5.2.** *Let $x$ be a rational number $x=\frac{d}{c}$ with $(d,c)=1$, $c\geqslant 1$. Then the function $D_v(s,x)$ of two complex parameters $s$ and $v$ is meromorphic over the whole of $\mathbb{C}^2$. If we fix $v$ such that $\Re v=1/2$ and $v\neq 1/2$, then $D_v(s,d/c)$ as a function of single variable $s$ has two simple poles at $s=v+1/2$ and $s=3/2-v$ with residues $c^{-2v}\zeta(2v)$ and $c^{2v-2}\zeta(2-2v)$, respectively, and it is regular elsewhere. Also it satisfies the functional equation*

$$D_v\left(s,\frac{d}{c}\right)=\left(\frac{4\pi}{c}\right)^{2s-1}\gamma(1-s,v)\left\{-\cos\pi s D_v\left(1-s,-\frac{a}{c}\right)+\sin\pi v D_v\left(1-s,\frac{a}{c}\right)\right\}, \quad (4.34)$$

*where* $ad \equiv \pmod{c}$ *and*

$$\gamma(u,v) = \frac{2^{2u-1}}{\pi}\Gamma(u+v-1/2)\Gamma(u-v+1/2). \tag{4.35}$$

*Proof.* See Lemma 3.7 of [28]. $\square$

Let

$$\hat{\phi}(s) = \int_0^\infty \phi(x)x^{s-1}dx$$

be the Mellin transform of $\phi$.

**Lemma 4.5.3.** *Assume that*

$$\int_0^\infty |g(x)|x^{-a}dx < \infty, \tag{4.36}$$

$$\int_{(a)} |\hat{\phi}(z)|dz < \infty. \tag{4.37}$$

*Then*

$$\frac{1}{2\pi i}\int_{(a)} \hat{\phi}(z)\hat{g}(1-z)dz = \int_0^\infty \phi(x)g(x)dx. \tag{4.38}$$

*Proof.* Consider

$$\begin{aligned}\frac{1}{2\pi i}\int_{(a)} \hat{\phi}(z)\hat{g}(1-z)dz &= \frac{1}{2\pi i}\int_{(a)} \hat{\phi}(z)\int_0^\infty g(x)x^{-z}dxdz \\ &= \frac{1}{2\pi i}\int_0^\infty g(x)\int_{(a)} \hat{\phi}(z)x^{-z}dzdx = \int_0^\infty \phi(x)g(x)dx.\end{aligned}$$

$\square$

**Theorem 4.5.4.** *(Theorem 5.2 of [27], page 89) Let $\phi$ be a smooth, compactly supported function on $\mathbb{R}^+$. Then for every $v$ with $\Re v = 1/2$, $(c,d)=1$, $c \geqslant 1$, one has*

$$\begin{aligned}\frac{4\pi}{c}\sum_{m\geqslant 1} e\left(\frac{md}{c}\right)\tau_v(m)\phi\left(\frac{4\pi\sqrt{m}}{c}\right) &= 2\frac{\zeta(2v)}{(4\pi)^{2v}}\hat{\phi}(2v+1) + 2\frac{\zeta(2-2v)}{(4\pi)^{2-2v}}\hat{\phi}(3-2v) \\ &+ \sum_{m\geqslant 1}\tau_v(m)\int_0^\infty \left[e\left(-\frac{ma}{c}\right)k_0(x\sqrt{m},v) + e\left(\frac{ma}{c}\right)k_1(x\sqrt{m},v)\right]\phi(x)xdx,\end{aligned} \tag{4.39}$$

*where* $ad \equiv 1 \pmod{c}$.

*Proof.* The requirements of the theorem imply that $\hat{\phi}(2s)$ is regular in $\sigma_0 \leqslant \Re s \leqslant \sigma_1$ for some $\sigma_0 < 0$ and $\sigma_1 > 1$. Then by the inverse Mellin transform

$$\phi\left(\frac{4\pi\sqrt{n}}{c}\right) = \frac{1}{i\pi}\int_{(b)} \hat{\phi}(2s)\left(\frac{c}{4\pi}\right)^{2s}\frac{1}{n^s}ds,\ 1 < b < \sigma_1.$$

Therefore,

$$\frac{4\pi}{c}\sum_{m\geqslant 1} e\left(\frac{md}{c}\right)\tau_v(m)\phi\left(\frac{4\pi\sqrt{m}}{c}\right) = \frac{1}{i\pi}\int_{(b)}\left(\frac{c}{4\pi}\right)^{2s-1} D_v\left(s,\frac{d}{c}\right)\hat{\phi}(2s)ds.$$

Note that the change of order of integration and summation in the last formula is justified by the absolute convergence. Moving the contour of integration to $\Re s = \delta$ with $\sigma_0 < \delta < 0$, we cross two simple poles of $D_v\left(s, \frac{d}{c}\right)$ at the points $v+1/2$ and $3/2-v$. Computation of residues gives the first two summands on the right-hand side of (4.39). We are left to evaluate

$$\frac{1}{i\pi}\int_{(\delta)}\left(\frac{c}{4\pi}\right)^{2s-1} D_v\left(s,\frac{d}{c}\right)\hat{\phi}(2s)ds.$$

Next, one can apply functional equation (4.34) and write the result in terms of Dirichlet series (4.33). Since $\Re(1-s) > 1$, the order of summation and integration can be changed. This yields

$$\sum_{m\geqslant 1}\tau_v(m)\left\{\frac{1}{i\pi}\int_{(\delta)}\gamma(1-s,v)\left(-e\left(-\frac{ma}{c}\right)\cos\pi s + e\left(\frac{ma}{c}\right)\sin\pi v\right)\hat{\phi}(2s)m^{s-1}ds\right\}.$$

Now we can move the contour of integration to $\Re s = \alpha$ such that $3/4 < \alpha < 1$. Then the result follows from Lemmas E.0.31 and 4.5.3. Indeed, let

$$g_1(x) := xk_0(x\sqrt{m}, v),$$

then

$$\hat{g_1}(1-2s) = -\gamma(1-s,v)\cos(\pi s)m^{s-1}$$

and

$$-\frac{1}{i\pi}\int_{(\alpha)}\gamma(1-s,v)\cos\pi s\hat{\phi}(2s)m^{s-1}ds=\int_0^\infty k_0(x\sqrt{m},v)\phi(x)xdx.$$

Note that condition (4.36) is satisfied for $g_1(x)$ if

$$\begin{cases}1-2\alpha>-1 & \text{as } x\to 0,\\ 1/2-2\alpha<-1 & \text{as } x\to\infty.\end{cases}$$

This explains our choice of $\alpha$. Analogously,

$$\frac{1}{i\pi}\int_{(\alpha)}\gamma(1-s,v)\sin\pi v\hat{\phi}(2s)m^{s-1}ds=\int_0^\infty k_1(x\sqrt{m},v)\phi(x)xdx.$$

$\square$

In order to apply Theorem 4.5.4, one has to exclude the coprimality condition in the sum. This can be done using the criterion of vanishing of classical Kloosterman sum (2.1.2).

Let $f(m,n,c):=F_{M,N}(m,n)J_{k-1}(\frac{4\pi\sqrt{mn}}{c})$.

**Proposition 4.5.5.** *Let $m,n,c$ be three strictly positive integers and $\rho$ be a prime number. Suppose $\rho^2$ divides $c$. Then*

$$\sum_{(q,mn)=1}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c)=$$

$$\sum_{m,n\geqslant 1}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c)$$

$$-\tau_{1/2+ir_2}(\rho)\sum_{m,n\geqslant 1}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n\rho,c)f(m,n\rho,c)$$

$$+\sum_{m,n\geqslant 1}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n\rho^2,c)f(m,n\rho^2,c).$$

*Proof.* Recall that $q=\rho^\nu$. Therefore,

$$\sum_{(q,mn)=1}=\sum_{\rho\nmid mn}=\sum_{m,n}-\sum_{\rho\mid mn}=\sum_{m,n}-\sum_{\rho\mid n}-\sum_{\rho\mid m,\rho\nmid n}.$$

The sum

$$\sum_{\rho|m,\rho\nmid n} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c) = 0$$

since the Kloosterman sum vanishes by Lemma 2.1.2. Further,

$$\sum_{\rho|n} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c) =$$

$$\sum_{n} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n\rho)S(m,n\rho,c)f(m,n\rho,c).$$

The identity (4.5) implies that

$$\tau_{1/2+ir_2}(n\rho) = \tau_{1/2+ir_2}(\rho)\tau_{1/2+ir_2}(n) - \tau_{1/2+ir_2}(\frac{n}{\rho}) \text{ if } (\rho,n)=\rho,$$

$$\tau_{1/2+ir_2}(n\rho) = \tau_{1/2+ir_2}(\rho)\tau_{1/2+ir_2}(n) \text{ if } (\rho,n)=1.$$

Finally,

$$\sum_{(q,mn)=1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c) =$$

$$\sum_{m,n\geqslant 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c)$$

$$- \tau_{1/2+ir_2}(\rho) \sum_{m,n\geqslant 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n\rho,c)f(m,n\rho,c)$$

$$+ \sum_{m,n\geqslant 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n\rho^2,c)f(m,n\rho^2,c).$$

□

## 4.6 Off-diagonal and off-off-diagonal terms

By proposition 4.5.5, the term (4.23) can be decomposed as follows

$$T_{M,N}(c) = TS(c,0) - \tau_{1/2+ir_2}(\rho)TS(c,1) + TS(c,2), \tag{4.40}$$

where

$$TS(c,B) = c\sum_{m,n\geqslant 1}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n\rho^B,c)f(m,n\rho^B,c) \text{ with } B=0,1,2 \quad (4.41)$$

and

$$f(m,n,c) = F_{M,N}(m,n)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \quad (4.42)$$

**Proposition 4.6.1.** *One has*

$$TS(c,B) = TS^*(c,B) + TS^+(c,B) + TS^-(c,B), \quad (4.43)$$

*where*

$$TS^*(c,B) = \sum_n \tau_{1/2+ir_2}(n)S(0,n\rho^B,c)\left[G^*_{r_1}(n\rho^B) + G^*_{-r_1}(n\rho^B)\right], \quad (4.44)$$

$$TS^{\mp}(c,B) = \sum_m\sum_n \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(0,n\rho^B\mp m,c)G^{\mp}_{r_1}(m,n\rho^B). \quad (4.45)$$

*Functions $G^*$, $G^-$, $G^+$ are defined as follows*

$$G^*_r(y) = \frac{\zeta(1+2ir)}{c^{2ir}}\int_0^\infty J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right)F_{M,N}(x,y)x^{ir}dx, \quad (4.46)$$

$$G^-_r(z,y) = 2\pi\int_0^\infty k_0\left(\frac{4\pi\sqrt{xz}}{c}, 1/2+ir\right)J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right)F_{M,N}(x,y)dx \quad (4.47)$$

*and*

$$G^+_r(z,y) = 2\pi\int_0^\infty k_1\left(\frac{4\pi\sqrt{xz}}{c}, 1/2+ir\right)J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right)F_{M,N}(x,y)dx. \quad (4.48)$$

*Proof.* The function $f$ is smooth, compactly supported, and thus satisfies all conditions of Theorem 4.5.4. Applying the summation formula (4.39) with $\phi(x) := f(\frac{c^2}{16\pi^2}x^2, n\rho^B, c)$, we obtain

$$\sum_{m\geqslant 1} e\left(\frac{md}{c}\right)\tau_{1/2+ir_1}(m)f(m,n\rho^B,c) =$$

$$\frac{\zeta(1+2ir_1)}{c^{1+2ir_1}}\int_0^\infty f(x,n\rho^B,c)x^{ir_1}dx + \frac{\zeta(1-2ir_1)}{c^{1-2ir_1}}\int_0^\infty f(x,n\rho^B,c)x^{-ir_1}dx$$

$$+\frac{2\pi}{c}\sum_{m\geqslant 1}\tau_{1/2+ir_1}(m)\int_0^\infty e\left(-\frac{m\overline{d}}{c}\right)k_0\left(\frac{4\pi}{c}\sqrt{xm},1/2+ir_1\right)f(x,n\rho^B,c)dx$$

$$+\frac{2\pi}{c}\sum_{m\geqslant 1}\tau_{1/2+ir_1}(m)\int_0^\infty e\left(\frac{m\overline{d}}{c}\right)k_1\left(\frac{4\pi}{c}\sqrt{xm},1/2+ir_1\right)f(x,n\rho^B,c)dx.$$

Finally, we plug this in (4.41) to get the required result. □

The first summand, including $TS^*(c)$, is an error term.

**Lemma 4.6.2.** *Let $l = 0, 1$. Then*

$$\sum_{\frac{q}{\rho^l}|c}\sum_{M,N\leqslant q^{1+\epsilon}} c^{-2}TS^*(c,B) \ll_{t_1,t_2,\epsilon,\rho} P(r_1)P(r_2)q^{|t_1|+|t_2|}q^{-1+\epsilon}. \tag{4.49}$$

*Proof.* We use (4.27) to estimate $F_{M,N}(m,n)$. $J$-Bessel function can be trivially bounded by 1. Then

$$G^*(n\rho^B) \ll_{t_1,t_2} P(r_1)P(r_2)\left(\frac{\hat{q}^2}{M}\right)^{|t_1|}\left(\frac{\hat{q}^2}{N}\right)^{|t_2|}\left(\frac{M}{N}\right)^{1/2}.$$

Since

$$S(0,n\rho^B,c) \ll (n\rho^B,c),$$

we have

$$TS^*(c,B) \ll_{t_1,t_2} (n\rho^B,c)(MN)^{1/2}q^\epsilon P(r_1)P(r_2)\left(\frac{\hat{q}^2}{M}\right)^{|t_1|}\left(\frac{\hat{q}^2}{N}\right)^{|t_2|}.$$

Therefore,

$$\sum_{\frac{q}{\rho^l}|c}\sum_{M,N\leqslant q^{1+\epsilon}} c^{-2}TS^*(c,B) \ll_{t_1,t_2,\epsilon,\rho} P(r_1)P(r_2)q^{|t_1|+|t_2|}q^{-1+\epsilon}.$$

□

The last two summands require more detailed treatment. We rewrite the sums $TS^{\pm}$ in the form that is more convenient for later computations.

$$TS^{-}(c,B)=\sum_{m}\sum_{n}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(0,n\rho^B-m,c)G^{-}_{r_1}(m,n\rho^B)$$

$$=\phi(c)\sum_{n}\tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n)G^{-}_{r_1}(n\rho^B,n\rho^B)+\sum_{h\neq 0}S(0,h,c)T^{-}_{h}(c,B)$$

and

$$TS^{+}(c,B)=\sum_{m}\sum_{n}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(0,n\rho^B+m,c)G^{+}_{r_1}(m,n\rho^B)$$

$$=\sum_{h\neq 0}S(0,h,c)T^{+}_{h}(c,B),$$

where

$$T^{\mp}_{h}(c,B)=\sum_{m\mp n\rho^B=h}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)G^{\mp}_{r_1}(m,\rho^B n). \tag{4.50}$$

At this point the non-diagonal term splits into off-diagonal (corresponds to $h=0$) and off-off-diagonal ($h\neq 0$) parts.

**Theorem 4.6.3.** *The following decomposition takes place*

$$M^{OD}=M^{OD}(0)-\tau_{1/2+ir_2}(\rho)M^{OD}(1)+M^{OD}(2), \tag{4.51}$$

$$M^{OOD}=M^{OOD}(0)-\tau_{1/2+ir_2}(\rho)M^{OOD}(1)+M^{OOD}(2). \tag{4.52}$$

*For $B=0,1,2$ we have*

$$M^{OD}(B)=2\pi i^{-k}\left(\sum_{\substack{q|c\\ c\ll C}}\frac{\phi(c)}{c^2}\sum_{M,N\ll q^{1+\epsilon}}\sum_{n}\tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n)G^{-}_{r_1}(n\rho^B,n\rho^B)\right.$$

$$\left.-1/\rho\sum_{\substack{\frac{q}{\rho}|c\\ c\ll C}}\frac{\phi(c)}{c^2}\sum_{M,N\ll q^{1+\epsilon}}\sum_{n}\tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n)G^{-}_{r_1}(n\rho^B,n\rho^B)\right),$$

$$M^{OOD}(B) = 2\pi i^{-k} \left( \sum_{\substack{q|c\\ c\ll C}} \frac{1}{c^2} \sum_{M,N\ll q^{1+\epsilon}} \sum_{h\neq 0} S(0,h,c)(T_h^-(c,B) + T_h^+(c,B)) \right.$$

$$\left. -1/\rho \sum_{\substack{\frac{q}{\rho}|c\\ c\ll C}} \frac{1}{c^2} \sum_{M,N\ll q^{1+\epsilon}} \sum_{h\neq 0} S(0,h,c)(T_h^-(c,B) + T_h^+(c,B)) \right).$$

*Here $T_h^{\pm}(c,B)$ is given by (4.50) and $G_r^{\pm}(z,y)$ by (4.48), (4.47).*

## 4.7 Extension of summations

Now we can reintroduce the summation over $c > C$ and $\max(M,N) \gg q^{1+\epsilon}$ for the off-diagonal term at the cost of admissible error.

**Proposition 4.7.1.** *For any $\epsilon > 0$*

$$\sum_{\substack{\frac{q}{\rho^l}|c\\ c>C}} \frac{\phi(c)}{c^2} \sum_{\max(M,N)\ll q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n) G_{r_1}^-(n\rho^B, n\rho^B) \ll_{\epsilon,\rho,t_1,t_2}$$

$$P(r_1)P(r_2)\hat{q}^{|t_1|+|t_2|+\epsilon} q^{-k/6+1/6}. \quad (4.53)$$

*Proof.* Let

$$\eta_C(c) = \begin{cases} 0 & \text{if } c > C \\ 1 & \text{if } c \leqslant C. \end{cases}$$

Consider

$$T_1 := \sum_{\substack{\frac{q}{\rho^l}|c\\ c>C}} \frac{\phi(c)}{c^2} \sum_{\max(M,N)\ll q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n) G_{r_1}^-(n\rho^B, n\rho^B) =$$

$$\sum_{\max(M,N)\ll q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n)$$

$$\times \int_0^\infty k_0\left(4\pi\sqrt{xn\rho^B}, 1/2+ir_1\right) J_{k-1}\left(4\pi\sqrt{xn\rho^B}\right) \sum_{\frac{q}{\rho^l}|c} (1-\eta_C(c))\phi(c) F_{M,N}(xc^2, n\rho^B)dx.$$

We use (4.27) to estimate $F_{M,N}(xc^2, n\rho^B)$, (C.7) to estimate $J_{k-1}\left(4\pi\sqrt{xn\rho^B}\right)$ and

$$k_0\left(4\pi\sqrt{xn\rho^B}, 1/2+ir_1\right) \ll 1.$$

Then

$$T_1 \ll_{\epsilon,t_1,t_2} P(r_1)P(r_2)\hat{q}^{|t_1|+|t_2|+\epsilon} \sum_{M,N\ll q^{1+\epsilon}} \frac{1}{\sqrt{MN}} \sum_{n\sim N} \int_0^{q^{-4/3}} (\sqrt{xn})^{k-1}\frac{M}{qx}dx$$
$$\ll_{\epsilon,\rho,t_1,t_2} P(r_1)P(r_2)\hat{q}^{|t_1|+|t_2|+\epsilon}q^{-k/6+1/6}.$$

□

**Proposition 4.7.2.** *For any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$*

$$\sum_{\frac{q}{\rho^l}|c} \frac{\phi(c)}{c^2} \sum_{\max(M,N)\gg q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n)G^-_{r_1}(n\rho^B, n\rho^B)$$
$$\ll_{\epsilon,\rho,A,t_1,t_2} P(r_1)P(r_2)q^{-A}. \quad (4.54)$$

*Proof.* It can be proved analogously to proposition 4.4.2. □

Now it is possible to combine all functions $F_M$ into $F$ and replace $\sum_{M,N} F_{M,N}$ by

$$F(x,y) := \frac{1}{\sqrt{xy}} W_{t_1,r_1}(\frac{x}{\hat{q}^2}) W_{t_2,r_2}(\frac{y}{\hat{q}^2}) F(x)F(y), \quad (4.55)$$

where $F(x)$ is a smooth function, compactly supported in $[1/2,\infty)$ such that $F(x) = 1$ for $x \geqslant 1$.

**Proposition 4.7.3.** *Up to an error term*

$$P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2+\epsilon}q^{-k/2},$$

*the product $F(x)F(y)$ can be replaced by 1 in (4.55).*

*Proof.* Consider

$$T_2 := \sum_{\substack{c \\ q|c}} \frac{\phi(c)}{c^2} \sum_n \tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n) \int_0^1 k_0\left(\frac{4\pi\sqrt{xn\rho^B}}{c}, 1/2+ir_1\right)$$

$$\times J_{k-1}\left(\frac{4\pi\sqrt{xn\rho^B}}{c}\right) \frac{1}{\sqrt{xn\rho^B}} W_{t_1,r_1}(\frac{x}{\hat{q}^2}) W_{t_2,r_2}(\frac{n\rho^B}{\hat{q}^2})(1-F(x))dx.$$

We estimate $k_0\left(\frac{4\pi\sqrt{xn\rho^B}}{c}, 1/2+ir_1\right)$ trivially by 1 and

$$J_{k-1}\left(\frac{4\pi\sqrt{xn\rho^B}}{c}\right) \ll \left(\frac{\sqrt{xn}}{c}\right)^{k-1}.$$

Suppose that $n < q$. Then $W_{t_2,r_2}$ can be estimated using (4.11). This gives

$$T_2 \ll_{t_1,t_2,\epsilon} P(r_1)P(r_2)q^{|t_1|+|t_2|+\epsilon}q^{-k/2}.$$

Suppose that $n \geqslant q$. We use (4.10), so that

$$T_2 \ll_{t_1,t_2,\epsilon} P(r_1)P(r_2)q^{|t_1|+\epsilon}q^{-k/2}.$$

$\square$

## 4.8 Asymptotics of the diagonal and off-diagonal terms

In this section we prove Theorems 4.0.4 and 4.0.6. Recall that

$$F(m,n) = \frac{1}{\sqrt{mn}} W_{t_1,r_1}\left(\frac{m}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{n}{\hat{q}^2}\right)$$

and

$$G^-_{r_1}(n\rho^B, n\rho^B) = 2\pi \int_0^\infty k_0\left(\frac{4\pi\sqrt{xn\rho^B}}{c}, 1/2+ir_1\right) J_{k-1}\left(\frac{4\pi\sqrt{xn\rho^B}}{c}\right) F(x, n\rho^B)dx.$$

Then the off-diagonal term can be written as

$$M^{OD}(B) = \hat{q}^{-2t_1-2t_2} \sum_n \frac{\tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n)}{n\rho^B} W_{t_2,r_2}\left(\frac{n\rho^B}{\hat{q}^2}\right) Z(n\rho^B),\ B = 0,1,2$$

with

$$Z = Z(n\rho^B) := 2\pi i^{-k} \int_0^\infty k_0(z, 1/2 + ir_1) J_{k-1}(z)$$

$$\times \left( \sum_{\substack{q|c \\ c \ll q^A}} \frac{\phi(c)}{c} W_{t_1,r_1} \left( \frac{z^2 c^2}{(4\pi)^2 \hat{q}^2 n\rho^B} \right) - \frac{1}{\rho} \sum_{\substack{\frac{q}{\rho}|c \\ c \ll q^A}} \frac{\phi(c)}{c} W_{t_1,r_1} \left( \frac{z^2 c^2}{(4\pi)^2 \hat{q}^2 n\rho^B} \right) \right) dz.$$

Note that we made a change of variables $x = \frac{z^2c^2}{(4\pi)^2 n\rho^B}$ in the integral. Applying (4.9), we have

$$Z = 2\pi i^{-k} \int_0^\infty k_0(z, 1/2 + ir_1) J_{k-1}(z) \frac{1}{2\pi i} \int_{(3)} G(s) \frac{\Gamma(s + ir_1 + k/2)\Gamma(s - ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2)\Gamma(t_1 - ir_1 + k/2)}$$

$$\times \zeta_q(1+2s) \left( \frac{z^2}{(4\pi)^2 \hat{q}^2 n\rho^B} \right)^{-s} \times \left[ \sum_{q|c} \frac{\phi(c)}{c^{1+2s}} - \frac{1}{\rho} \sum_{\frac{q}{\rho}|c} \frac{\phi(c)}{c^{1+2s}} \right] \frac{2s ds}{s^2 - t_1^2} dz.$$

The term in the brackets can be simplified. Consider

$$\sum_{q|c} \frac{\phi(c)}{c^{1+2s}} = \sum_{k \geqslant \nu} \frac{\phi(\rho^k)}{\rho^{k(1+2s)}} \sum_{(c,\rho)=1} \frac{\phi(c)}{c^{1+2s}}$$

$$= \frac{\phi(q)}{q} \sum_{k \geqslant \nu} \frac{1}{(\rho^{2s})^k} \sum_{(c,\rho)=1} \frac{\phi(c)}{c^{1+2s}} = \frac{\phi(q)}{q^{1+2s}} \frac{1}{1-\rho^{-2s}} \frac{\zeta_q(2s)}{\zeta_q(2s+1)}.$$

Analogously,

$$\sum_{\frac{q}{\rho}|c} \frac{\phi(c)}{c^{1+2s}} = \frac{\phi(q)}{q^{1+2s}} \frac{\rho^{2s}}{1-\rho^{-2s}} \frac{\zeta_q(2s)}{\zeta_q(2s+1)}.$$

Combining the last two formulas,

$$\sum_{q|c} \frac{\phi(c)}{c^{1+2s}} - \frac{1}{\rho} \sum_{\frac{q}{\rho}|c} \frac{\phi(c)}{c^{1+2s}} = \frac{\phi(q)}{q^{1+2s}} \frac{1-\rho^{2s-1}}{1-\rho^{-2s}} \frac{\zeta_q(2s)}{\zeta_q(2s+1)}.$$

Lemma C.0.19 implies that

$$\int_0^\infty k_0(z,1/2+ir_1)J_{k-1}(z)z^{-2s}dz = \frac{\Gamma(2s)}{2^{2s+1}\cos(\pi(1/2+ir_1))}$$
$$\times \frac{\Gamma(ir_1+k/2-s)\Gamma(-ir_1-k/2+s+1)-\Gamma(-ir_1+k/2-s)\Gamma(ir_1-k/2+s+1)}{\Gamma(-ir_1+k/2+s)\Gamma(ir_1+k/2+s)\Gamma(ir_1-k/2+s+1)\Gamma(-ir_1-k/2+s+1)}.$$

By duplication formula (A.10) and reflection formula (A.9),

$$\int_0^\infty k_0(z,1/2+ir_1)J_{k-1}(z)z^{-2s}dz = -\frac{i^k\Gamma(s)\Gamma(s+1/2)}{2^2\pi^{3/2}\sin(\pi ir_1)}$$
$$\times \frac{\Gamma(ir_1+k/2-s)\Gamma(-ir_1+k/2-s)}{\Gamma(ir_1+k/2+s)\Gamma(-ir_1+k/2+s)}[\sin(\pi(-s-ir_1))-\sin(\pi(-s+ir_1))].$$

Note that

$$\frac{\Gamma(1/2-s)\Gamma(1/2+s)}{2\pi\sin(\pi ir_1)}[\sin(\pi(-s-ir_1))-\sin(\pi(-s+ir_1))] = -1. \tag{4.56}$$

Consequently,

$$Z = \frac{\phi(q)}{q}\frac{1}{2\pi i}\int_{(3)} G(s)\frac{\Gamma(-s+ir_1+k/2)\Gamma(-s-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\zeta_q(1-2s)\left(\frac{n\rho^B}{\hat{q}^2}\right)^s\frac{2sds}{s^2-t_1^2}.$$

Shifting the contour of integration to $\Re(s)=-3$, we cross poles at $s=\pm t_1$. Hence

$$Z = -\frac{\phi(q)}{q}\frac{1}{2\pi i}\int_{(3)} G(s)\frac{\Gamma(s+ir_1+k/2)\Gamma(s-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\zeta_q(1+2s)\left(\frac{n\rho^B}{\hat{q}^2}\right)^{-s}\frac{2sds}{s^2-t_1^2}$$
$$+\frac{\phi(q)}{q}\sum_{\epsilon_1=\pm1}\frac{\Gamma(\epsilon_1t_1+ir_1+k/2)\Gamma(\epsilon_1t_1-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\zeta_q(1+2\epsilon_1t_1)\left(\frac{n\rho^B}{\hat{q}^2}\right)^{-\epsilon_1t_1}.$$

Substitution of $Z$ into $M^{OD}(B)$ gives

$$M^{OD}(B) = \frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2}\sum_n\frac{\tau_{1/2+ir_1}(n\rho^B)\tau_{1/2+ir_2}(n)}{n\rho^B}W_{t_2,r_2}(\frac{n\rho^B}{\hat{q}^2})\Bigg(-W_{t_1,r_1}(\frac{n\rho^B}{\hat{q}^2})$$
$$+\sum_{\epsilon_1=\pm1}\frac{\Gamma(\epsilon_1t_1+ir_1+k/2)\Gamma(\epsilon_1t_1-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\zeta_q(1+2\epsilon_1t_1)\left(\frac{n\rho^B}{\hat{q}^2}\right)^{-\epsilon_1t_1}\Bigg).$$

The property of multiplicity (4.5) implies that

$$\sum_{\substack{n\geqslant 1\\(n,\rho)=1}} \tau_{1/2+ir_2}(n)f(n) =$$

$$\sum_{n\geqslant 1} \tau_{1/2+ir_2}(n)f(n) - \tau_{1/2+ir_2}(\rho)\sum_{n\geqslant 1} \tau_{1/2+ir_2}(n)f(n\rho) + \sum_{n\geqslant 1} \tau_{1/2+ir_2}(n)f(n\rho^2).$$

Thus,

$$M^D + M^{OD} = \frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2}\sum_{(n,\rho)=1}\frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n}W_{t_2,r_2}(\frac{n}{\hat{q}^2})$$
$$\times\left(\sum_{\epsilon_1=\pm 1}\frac{\Gamma(\epsilon_1 t_1+ir_1+k/2)\Gamma(\epsilon_1 t_1-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\zeta_q(1+2\epsilon_1 t_1)\left(\frac{n}{\hat{q}^2}\right)^{-\epsilon_1 t_1}\right). \quad (4.57)$$

Ramanujan's identity (4.6) gives

$$\sum_{(n,\rho)=1}\frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n^{1+\epsilon_1 t_1+s}} = \zeta_q(1+\epsilon_1 t_1+s+ir_1-ir_2)\zeta_q(1+\epsilon_1 t_1+s-ir_1+ir_2)$$
$$\times\frac{\zeta_q(1+\epsilon_1 t_1+s+ir_1+ir_2)\zeta_q(1+\epsilon_1 t_1+s-ir_1-ir_2)}{\zeta_q(2+2\epsilon_1 t_1+2s)}.$$

Therefore,

$$M^D + M^{OD} = \frac{\phi(q)}{q}\sum_{\epsilon_1=\pm 1}\frac{\Gamma(\epsilon_1 t_1+ir_1+k/2)\Gamma(\epsilon_1 t_1-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\zeta_q(1+2\epsilon_1 t_1)$$
$$\times\hat{q}^{-2t_1-2t_2+2\epsilon_1 t_1}\frac{1}{2\pi i}\int_{\Re s=3}\frac{G(s)}{G(t_2)}\hat{q}^{2s}\zeta_q(1+2s)\frac{\Gamma(s+ir_2+k/2)\Gamma(s-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}$$
$$\times\frac{\prod\zeta_q(1+\epsilon_1 t_1+s\pm ir_1\pm ir_2)}{\zeta_q(2+2\epsilon_1 t_1+2s)}\frac{2sds}{s^2-t_2^2}.$$

Shifting the contour of integration to $\Re s = -1/2$, the resulting integral is bounded by $P(r_1)P(r_2)q^{(\epsilon_1-1)t_1-t_2}q^{-1/2}$ plus the contribution of simple poles at $s=\pm t_2$. Up to an error term,

$$M^D + M^{OD} = \frac{\phi(q)}{q} \sum_{\epsilon_1,\epsilon_2=\pm 1} \hat{q}^{-2t_1-2t_2+2\epsilon_1 t_1+2\epsilon_2 t_2}$$

$$\times \zeta_q(1+2\epsilon_1 t_1)\zeta_q(1+2\epsilon_2 t_2)\frac{\prod \zeta_q(1+\epsilon_1 t_1+\epsilon_2 t_2 \pm ir_1 \pm ir_2)}{\zeta_q(2+2\epsilon_1 t_1+2\epsilon_2 t_2)}$$

$$\times \frac{\Gamma(\epsilon_1 t_1+ir_1+k/2)\Gamma(\epsilon_1 t_1-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\frac{\Gamma(\epsilon_2 t_2+ir_2+k/2)\Gamma(\epsilon_2 t_2-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}.$$

By letting shifts tend to zero in (4.57), we find

$$M^D + M^{OD} = \left(\frac{\phi(q)}{q}\right)^2 \sum_{(n,\rho)=1} \frac{\tau(n)^2}{n} W_{0,0}(\frac{n}{\hat{q}^2}) \log\left(\frac{\hat{q}^2}{n}\right). \tag{4.58}$$

The equality (4.7) gives

$$M^D + M^{OD} = \frac{1}{2\pi i}\left(\frac{\phi(q)}{q}\right)^2 \int_{(3)} \frac{G(s)}{G(0)}\frac{\Gamma(k/2+s)^2}{\Gamma(k/2)^2}\zeta_q(1+2s)\hat{q}^{2s}\frac{\zeta_q(1+s)^4}{\zeta_q(2+2s)}$$

$$\times \left[\log \hat{q}^2 + 4\frac{\zeta_q'}{\zeta_q}(1+s) - 2\frac{\zeta_q'}{\zeta_q}(2+2s)\right]\frac{2ds}{s}.$$

Shifting the contour of integration to $\Re s = -1/2$, the resulting integral is bounded by $q^{-1/2}$ plus the contribution of multiple poles at $s=0$. Calculation of the residue

$$\left(\frac{\phi(q)}{q}\right)^7 \frac{1}{\zeta_q(2)} \operatorname{Res}_{s=0} \frac{\hat{q}^{2s}}{s^6}\left(\log \hat{q} - \frac{4}{s}\right)$$

shows that the main term is

$$(\frac{\phi(q)}{q})^7 \frac{\rho^2}{\rho^2-1}\frac{(\log q)^6}{60\pi^2}.$$

# 5

# The fourth moment: off-off-diagonal term

In this chapter, we study the off-off-diagonal part of the fourth moment and find its contribution to the main term of asymptotic formula.

**Theorem 5.0.1.** *For any $\epsilon > 0$, up to an error*

$$O_{\epsilon,\rho,t_1,t_2}(P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2}q^{\epsilon}(q^{-\frac{2k-3}{12}}+q^{-1/4})),$$

*we have*

$$M^{OOD} = \frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm 1}\frac{\zeta_q(1+2i\epsilon_1 r_1)\zeta_q(1+2i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}\hat{q}^{-2t_1-2t_2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}$$

$$\times\, \zeta_q(1+t_1+t_2+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1+t_2-t_1+i\epsilon_1 r_1+i\epsilon_2 r_2)$$

$$\times\zeta_q(1+t_1-t_2+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1-t_1-t_2+i\epsilon_1 r_1+i\epsilon_2 r_2)\frac{\Gamma(k/2-t_1+i\epsilon_1 r_1)\Gamma(k/2-t_2+i\epsilon_2 r_2)}{\Gamma(k/2+t_1-i\epsilon_1 r_1)\Gamma(k/2+t_2-i\epsilon_2 r_2)}.$$

**Remark 5.0.2.** *The biggest error term appears in Lemma 5.3.1.*

## 5.1 Quadratic divisor problem

The off-off-diagonal term can be treated by $\delta$-symbol method. In [12] Duke, Friedlander and Iwaniec proved Theorem 5.1.1 using Jutila's summation formula (4.30).

**Theorem 5.1.1.** *(Duke, Friedlander, Iwaniec)*

*Let $a, b \geqslant 1$, $(a, b) = 1$, $h \neq 0$. Let*

$$D_f(a, b; h) = \sum_{am \mp bn = h} \tau(m)\tau(n) f(am, bn)$$

*with*

$$x^i y^j f^{(ij)}(x, y) \ll (1 + \frac{x}{X})^{-1}(1 + \frac{y}{Y})^{-1} P^{i+j}. \tag{5.1}$$

*Assume that*

$$ab < P^{-5/4}(X + Y)^{-5/4}(XY)^{1/4+\epsilon}. \tag{5.2}$$

*Then*

$$D_f(a, b; h) = \int_0^\infty g(x, \pm x \mp h) dx + O(P^{5/4}(X + Y)^{1/4}(XY)^{1/4+\epsilon}).$$

*Here $g(x, y) = f(x, y)\Lambda_{a,b,h}(x, y)$ with*

$$\Lambda_{a,b,h}(x, y) = \frac{1}{ab} \sum_{w=1}^{\infty} w^{-2}(ab, w) S(0, h, w)(\log x - \lambda_{aw})(\log y - \lambda_{bw}),$$

$$\lambda_{aw} = -2\gamma + \log(\frac{aw^2}{(a, w)^2}).$$

Applying formula (4.39), we generalize Theorem 5.1.1 as follows.

**Theorem 5.1.2.** *Let $a, b \geqslant 1$, $(a, b) = 1$, $h \neq 0$. Let*

$$D_f(a, b; h) = \sum_{am \mp bn = h} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) f(am, bn)$$

*with*

$$x^i y^j f^{(ij)}(x, y) \ll (1 + \frac{x}{X})^{-1}(1 + \frac{y}{Y})^{-1} P^{i+j}. \tag{5.3}$$

*Assume that*

$$ab < P^{-5/4}(X + Y)^{-5/4}(XY)^{1/4+\epsilon}. \tag{5.4}$$

*Then*

$$D_f(a,b;h) = \int_0^\infty g(x, \pm x \mp h)dx + O(P(r_1)P(r_2)P^{5/4}(X+Y)^{1/4}(XY)^{1/4+\epsilon}).$$

*Here* $g(x,y) = f(x,y)\Lambda_{a,b,h}(x,y)$ *with*

$$\Lambda_{a,b,h}(x,y) := \sum_{w=1}^{\infty} S(0,h,w) \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{(ab,w)(a,w)^{2i\epsilon_1 r_1}(b,w)^{2i\epsilon_2 r_2}}{a^{1+i\epsilon_1 r_1} b^{1+i\epsilon_2 r_2} w^{2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}}$$

$$\times \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2) x^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}. \quad (5.5)$$

### 5.1.1 *Preliminary results*

We follow the proof of [12]. For any $n \in \mathbb{Z}$ we define

$$\delta(n) := \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases} \quad (5.6)$$

Let $\omega(u)$ be a smooth compactly supported function on $\mathbb{R}$ such that

- $\omega(0) = 0$,
- $\omega(u) = \omega(-u)$,
- $\sum_{w=1}^{\infty} \omega(w) = 1$,
- $\omega(u)$ is supported in $Q \leqslant u \leqslant 2Q$,
- $\omega^{(j)} \ll Q^{-j-1}$, $j \geqslant 0$.

Then

$$\delta(n) = \sum_{w|n} \left(\omega(w) - \omega\left(\frac{n}{w}\right)\right) = \sum_{w=1}^{\infty} \sum_{\substack{d \pmod w \\ (d,w)=1}} e\left(\frac{dn}{w}\right) \Delta_w(n), \quad (5.7)$$

where

$$\Delta_w(u) := \sum_{r=1}^{\infty} (wr)^{-1} \left(\omega(wr) - \omega\left(\frac{u}{wr}\right)\right). \quad (5.8)$$

**Lemma 5.1.3.** *([12], p.213) For $j \geqslant 1$ we have*

$$\int_{-\infty}^{+\infty} f(u)\Delta_w(u)du = f(0) + O\left(Q^{-1}w^j \int (Q^{-j}|f(u)| + Q^j|f^{(j)}(u)|)du\right). \tag{5.9}$$

**Lemma 5.1.4.** *([12], p.213)*

$$\Delta_w(u) \ll (wQ + Q^2)^{-1} + (wQ + |u|)^{-1}. \tag{5.10}$$

Assume that the function $f$ is compactly supported in $[X, 2X] \times [Y, 2Y]$ and it satisfies (5.3). Let $\phi(u)$ be a smooth function supported on $|u| < U$ such that $\phi(0) = 1$ and $\phi^{(i)} \ll U^{-i}$. Suppose $U \leqslant P^{-1}\min(X, Y)$, then $F(x, y) := f(x, y)\phi(x - y - h)$ satisfies

$$F^{(ij)}(x, y) \ll \left(\frac{1}{U} + \frac{P}{X}\right)^i \left(\frac{1}{U} + \frac{P}{Y}\right)^j \ll U^{-i-j}. \tag{5.11}$$

We choose $Q = U^{1/2}$ so that $\Delta_w(u) = 0$ if $|u| \leqslant U$ and $w \geqslant 2Q$. Let

$$E(x, y) = F(ax, by)\Delta_w(ax - by - h). \tag{5.12}$$

**Lemma 5.1.5.** *For all $i, j \geqslant 0$ we have*

$$E^{(ij)}(x, y) \ll \frac{a^i b^j}{wQ}\left(\frac{1}{wQ}\right)^{i+j}. \tag{5.13}$$

*Proof.* Consider

$$\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j}E(x, y) \ll \sum_{k=0}^{i}\sum_{n=0}^{j} a^{i-k}b^{j-n}\, \frac{\partial^{i-k}}{\partial z^{i-k}}\frac{\partial^{j-n}}{\partial t^{j-n}}F(z, t)\bigg|_{\substack{z=ax\\ t=by}} \times a^k b^n \frac{\partial^{k+n}}{\partial z^{k+n}}\Delta_w(z)\bigg|_{z=ax-by+h}.$$

By (5.11),

$$\frac{\partial^{i-k}}{\partial z^{i-k}}\frac{\partial^{j-n}}{\partial t^{j-n}}F(z, t) \ll U^{k-i+n-j}.$$

Using definition (5.8), we have

$$\frac{\partial^{k+n}}{\partial z^{k+n}}\Delta_w(z) = -\sum_{r=1}^{\infty}\frac{1}{(wr)^{k+n+1}}w^{(k+n)}\left(\frac{z}{wr}\right) \ll \sum_{\substack{r\geqslant 1\\ z/wr\sim Q}}\frac{1}{(wr)^{k+n+1}}\frac{1}{Q^{k+n+1}} \ll \frac{1}{(wQ)^{k+n+1}}.$$

Note that

$$U = Q^2 > wQ.$$

Therefore,

$$\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j}E(x,y) \ll \frac{a^i b^j}{(wQ)^{i+j+1}}.$$

□

**Lemma 5.1.6.** *([12], p.216) We have*

$$\int_0^\infty \int_0^\infty E(x,y)dxdy \ll (ab)^{-1}(X+Y)^{-1}XY\log Q. \tag{5.14}$$

*5.1.2 Proof of Theorem 5.1.2*

We prove the case $am - bn = h$.

$$\begin{aligned} D_f(a,b,h) &= D_F(a,b;h) \\ &= \sum_{1\leqslant w<2Q}\ \sum_{\substack{d \pmod w \\ (d,w)=1}} e\left(\frac{-dh}{w}\right)\sum_m\sum_n \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)e\left(\frac{dam-dbn}{w}\right)E(m,n). \end{aligned} \tag{5.15}$$

According to Theorem 4.5.4,

$$\begin{aligned} &\sum_{m\geqslant 1} e\left(\frac{md}{w}\right)\tau_{1/2+ir_1}(m)E(m,n) = \\ &\qquad \frac{\zeta(1+2ir_1)}{w^{1+2ir_1}}\int_0^\infty E(x,n)x^{ir_1}dx + \frac{\zeta(1-2ir_1)}{w^{1-2ir_1}}\int_0^\infty E(x,n)x^{-ir_1}dx \\ &\qquad + \frac{2\pi}{w}\sum_{m\geqslant 1}\tau_{1/2+ir_1}(m)\int_0^\infty e\left(-\frac{m\overline{d}}{w}\right)k_0\left(\frac{4\pi}{w}\sqrt{xm}, 1/2+ir_1\right)E(x,n)dx \\ &\qquad + \frac{2\pi}{w}\sum_{m\geqslant 1}\tau_{1/2+ir_1}(m)\int_0^\infty e\left(\frac{m\overline{d}}{w}\right)k_1\left(\frac{4\pi}{w}\sqrt{xm}, 1/2+ir_1\right)E(x,n)dx \end{aligned}$$

with

$$k_0(x,v) = \frac{1}{2\cos\pi v}(J_{2v-1}(x) - J_{1-2v}(x)),$$

$$k_1(x,v) = \frac{2}{\pi} \sin \pi v K_{2v-1}(x).$$

Applying the summation formula for both variables $m$ and $n$, we have

$$\frac{w^2}{(ab,w)} \sum_m \sum_n \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) e\left(\frac{dam - dbn}{w}\right) E(m,n) = I$$

$$+ \sum_{m=1}^{\infty} \tau_{1/2+ir_1}(m) e\left(-m\frac{\overline{ad}}{w}\right) I_a(m) + + \sum_{n=1}^{\infty} \tau_{1/2+ir_2}(n) e\left(n\frac{\overline{bd}}{w}\right) I_b(n)$$

$$+ \sum_m \sum_n \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) e\left(-m\frac{\overline{ad}}{w} + n\frac{\overline{bd}}{w}\right) I_{ab}(m,n) + *****,$$

where

$$I = \sum_{\epsilon_1,\epsilon_2=\pm1} \frac{(a,w)^{2i\epsilon_1 r_1}(b,w)^{2i\epsilon_2 r_2}}{w^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)$$
$$\times \int_0^\infty \int_0^\infty E(x,y) x^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2} dxdy; \quad (5.16)$$

$$I_a(m) = 2\pi \sum_{\epsilon_2=\pm1} \frac{(b,w)^{2i\epsilon_2 r_2}}{w^{2i\epsilon_2 r_2}} \zeta(1+2i\epsilon_2 r_2)$$
$$\times \int_0^\infty \int_0^\infty k_0\left(\frac{4\pi(a,w)}{w}\sqrt{xm}, 1/2+ir_1\right) E(x,y) y^{i\epsilon_2 r_2} dxdy; \quad (5.17)$$

$$I_b(n) = 2\pi \sum_{\epsilon_1=\pm1} \frac{(a,w)^{2i\epsilon_1 r_1}}{w^{2i\epsilon_1 r_1}} \zeta(1+2i\epsilon_1 r_1)$$
$$\times \int_0^\infty \int_0^\infty k_0\left(\frac{4\pi(b,w)}{w}\sqrt{xn}, 1/2+ir_2\right) E(x,y) x^{i\epsilon_1 r_1} dxdy; \quad (5.18)$$

$$I_{ab}(m,n) = 4\pi^2 \int_0^\infty k_0\left(\frac{4\pi(a,w)}{w}\sqrt{xm}, 1/2+ir_1\right)$$
$$\times k_0\left(\frac{4\pi(b,w)}{w}\sqrt{xn}, 1/2+ir_2\right) E(x,y) dxdy \quad (5.19)$$

and $*****$ are the five other terms with the $k_1$ Bessel kernel.

Therefore,

$$D_F(a,b,h) = \sum_{w<2Q} \frac{(ab,w)}{w^2} \Bigg[ S(0,h,w)I + \sum_{m\geqslant 1} \tau_{1/2+ir_1}(m) S(h,\bar{a}m,w) I_a(m)$$
$$+ \sum_{n\geqslant 1} \tau_{1/2+ir_2}(n) S(h,-\bar{b}n,w) I_b(n)$$
$$+ \sum_{m\geqslant 1}\sum_{n\geqslant 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) S(h,\bar{a}m-\bar{b}n,w) I_{ab}(m,n) + ***** \Bigg]. \quad (5.20)$$

Contribution to the main term is given by the integral $I$. Let

$$C(x,y) := \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{(a,w)^{2i\epsilon_1 r_1}(b,w)^{2i\epsilon_2 r_2}}{a^{1+i\epsilon_1 r_1} b^{1+i\epsilon_2 r_2} w^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}}$$
$$\times \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2) F(x,y) x^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}. \quad (5.21)$$

Then

$$I = \int_0^\infty \int_0^\infty C(x,y)\Delta_w(x-y-h)dxdy = \int_0^\infty \int_0^\infty C(x,x-h+u)\Delta_w(u)dudx. \quad (5.22)$$

Lemma 5.1.3 implies that

$$\int_0^\infty C(x,x-h+u)\Delta_w(u)du = C(x,x-h) + O\left(\frac{1}{ab}P(r_1)P(r_2)\left(\frac{w}{Q}\right)^j\right). \quad (5.23)$$

If $w \leqslant Q^{1-\epsilon}$, we can take $j$ sufficiently large, so that

$$I = \int C(x,x-h)dx + O\left(\frac{1}{ab}P(r_1)P(r_2)(Q)^{-\Omega}\right) \text{ for any } \Omega > 0. \quad (5.24)$$

If $w > Q^{1-\epsilon}$, then we apply the bound (5.14), which is valid for any $w$

$$I \ll P(r_1)P(r_2)(ab)^{-1}(X+Y)^{-1}XY\log Q. \quad (5.25)$$

Finally, the contribution of $I$ integral to $D_f(a,b,h)$ is

$$\sum_{w=1}^{\infty}\frac{(ab,w)}{w^2}S(h,0,w)\int_0^{\infty}C(x,x-h)dx+O\left(P(r_1)P(r_2)\frac{1}{ab}\frac{XY}{X+Y}Q^{-1+\epsilon}\right). \tag{5.26}$$

The integrals $I_a(m)$, $I_b(n)$, $I_{ab}(m,n)$ and the five other term $*****$ contribute to $D_f(a,b,h)$ as an error.

Consider $I_a(m)$. If $m \geqslant \frac{aX}{(a,w)^2}Q^{-2+\epsilon}$, we integrate $j$ times by parts in $x$ using Lemma C.0.21.

$$I_a(m) \ll P(r_1)P(r_2)\left(\frac{w}{(a,w)\sqrt{m}}\right)^j$$
$$\times\int_{ax\sim X}\frac{\partial^j}{\partial x^j}(E(x,y)x^{-ir_1})x^{ir_1+j/2}J_{j+2ir}\left(\frac{4\pi(a,w)\sqrt{xm}}{w}\right)dx.$$

We estimate $J$-Bessel function by 1 and use (5.13)

$$I_a(m) \ll P(r_1)P(r_2)\frac{1}{wQ}\left(\frac{aX}{(a,w)^2m}Q^{-2+\epsilon}\right)^{j/2}.$$

Thus, the $I_a(m)$ can be made arbitrary small if $m \geqslant \frac{aX}{(a,w)^2}Q^{-2+\epsilon}$. Analogously, $I_b(n)$ is small if $n \geqslant \frac{bX}{(b,w)^2}Q^{-2+\epsilon}$ and $I_{ab}(m,n)$ is small if $m \geqslant \frac{aX}{(a,w)^2}Q^{-2+\epsilon}$ and $n \geqslant \frac{bX}{(b,w)^2}Q^{-2+\epsilon}$.

In the range $m < \frac{aX}{(a,w)^2}Q^{-2+\epsilon}$ and $n < \frac{bX}{(b,w)^2}Q^{-2+\epsilon}$, we use (5.14) and (C.7)

$$I_a(m) \ll P(r_1)P(r_2)\left(\frac{aw^2(a,w)^2}{Xm}\right)^{1/4}(ab)^{-1}\frac{XY}{X+Y}Q^{\epsilon},$$

$$I_b(n) \ll P(r_1)P(r_2)\left(\frac{bw^2(b,w)^2}{Xm}\right)^{1/4}(ab)^{-1}\frac{XY}{X+Y}Q^{\epsilon},$$

$$I_{ab}(m,n) \ll P(r_1)P(r_2)\left(\frac{abw^4(a,w)^4(b,w)^4}{XYmn}\right)^{1/4}(ab)^{-1}\frac{XY}{X+Y}Q^{\epsilon}.$$

Summing over $m,n$ in the given range, we have

$$\sum_{m<\frac{aX}{(a,w)^2}Q^{-2+\epsilon}}\tau_{1/2+ir_1}(m)|I_a(m)| \ll \frac{w^{1/2}}{b}\frac{X^{3/2}Y}{X+Y}Q^{-3/2+\epsilon},$$

$$\sum_{n<\frac{bX}{(b,w)^2}Q^{-2+\epsilon}} \tau_{1/2+ir_2}(n)|I_b(m)| \ll \frac{w^{1/2}}{a}\frac{Y^{3/2}X}{X+Y}Q^{-3/2+\epsilon},$$

$$\sum_{m<\frac{aX}{(a,w)^2}Q^{-2+\epsilon}} \sum_{n<\frac{bX}{(b,w)^2}Q^{-2+\epsilon}} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_1}(n)|I_{ab}(m,n)| \ll w\frac{(XY)^{3/2}}{X+Y}Q^{-3+\epsilon}.$$

Finally, we use Weil's bound (2.7) for Kloosterman sums and the following bound for Ramanujan sums

$$S(h,0,q) \ll (h,q).$$

Then $I_a(m)$, $I_b(n)$ and $I_{ab}(m,n)$ contribute to $D_f(a,b;h)$ as

$$P(r_1)P(r_2)\frac{(XY)^{3/2}}{X+Y}Q^{-5/2+\epsilon}.$$

The total error term is

$$P(r_1)P(r_2)\left((ab)^{-1}\frac{XY}{X+Y}Q^{-1+\epsilon} + \frac{(XY)^{3/2}}{X+Y}Q^{-5/2+\epsilon}\right).$$

Taking $U = Q^2 = P^{-1}(X+Y)^{-1}XY$, we obtain the required result.

## 5.2 Estimation of $G^{\mp}_{r_1}$

In order to apply Theorem 5.1.2 to

$$T_h^{\pm}(c,B) = \sum_{m\pm n\rho^B=h} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)G^{\pm}_{r_1}(m,\rho^B n),$$

we show that the functions $G^{\pm}_{r_1}$, defined by (4.47) and (4.48), satisfy condition (5.3).

Let $Q := 1 + \frac{\sqrt{MN}}{c}$, $Z := \frac{Q^2c^2}{M}$, $Y := N$.

**Lemma 5.2.1.** *For all positive $n_1$ and $n_2$*

$$z^j y^i \frac{\partial^j}{\partial z^j}\frac{\partial^i}{\partial y^i}G^{\pm}_{r_1}(z,y) \ll_{t_1,t_2}$$

$$(1+\frac{z}{Z})^{-n_1}(1+\frac{y}{Y})^{-n_2}P(r_1)P(r_2)q^{|t_1|+|t_2|}\frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1}Q^{j+i-k+1/2}. \quad (5.27)$$

*Proof.* Consider

$$G^{-}_{r_1}(z,y) = 2\pi \int_0^\infty k_0\left(\frac{4\pi\sqrt{xz}}{c}, 1/2+ir_1\right) J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right) F_{M,N}(x,y)dx$$

$$= -\frac{\pi}{\sin \pi i r_1} \int_0^\infty \left[J_{2ir_1}\left(\frac{4\pi\sqrt{xz}}{c}\right) - J_{-2ir_1}\left(\frac{4\pi\sqrt{xz}}{c}\right)\right] J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right) F_{M,N}(x,y)dx.$$

Suppose that $z > Z$. Put $u := \frac{4\pi\sqrt{xz}}{c}$, then

$$G^{-}_{r_1}(z,y) = -\frac{c^2}{8\pi z \sin \pi i r_1} \int_0^\infty u \left[J_{2ir_1}(u) - J_{-2ir_1}(u)\right] J_{k-1}\left(u\sqrt{\frac{y}{z}}\right) F_{M,N}\left(\frac{c^2u^2}{16\pi^2 z}, y\right) du.$$

It is sufficient to estimate

$$G_1(z,y) := -\frac{c^2}{8\pi z \sin \pi i r_1} \int_0^\infty u J_{2ir_1}(u) J_{k-1}\left(u\sqrt{\frac{y}{z}}\right) F_{M,N}\left(\frac{c^2u^2}{16\pi^2 z}, y\right) du.$$

Note that $F_{M,N}(x,y)$ is compactly supported on $[M/2, 3M] \times [N/2, 3N]$. Let

$$f(u) := g_1(u) g_2(u) u^{-2ir_1}$$

with

$$g_1(u) := J_{k-1}\left(u\sqrt{\frac{y}{z}}\right) \text{ and } g_2(u) := F_{M,N}\left(\frac{c^2u^2}{16\pi^2 z}, y\right).$$

The recurrent relation (C.4) implies that

$$G_1(z,y) = -\frac{c^2}{8\pi z \sin \pi i r_1} \int_0^\infty \left(u^{1+2ir_1} J_{1+2ir_1}(u)\right)' f(u) du.$$

Integration by parts gives

$$G_1(z,y) = \frac{c^2}{8\pi z \sin \pi i r_1} \int_0^\infty u^{1+2ir_1} J_{1+2ir_1}(u) f'(u) du =$$

$$- \frac{c^2}{8\pi z \sin \pi i r_1} \int_0^\infty u^{2+2ir_1} J_{2+2ir_1}(u) (\frac{1}{u} f'(u))' du.$$

Repeating the procedure $n$ times, we have

$$G_1(z,y) = (-1)^{n+1}\frac{c^2}{8\pi z\sin \pi i r_1}\int_0^\infty u^{n+2ir_1}J_{n+2ir_1}(u)h_n(u)du =$$

$$(-1)^{n+1}\frac{c^2}{8\pi z\sin \pi i r_1}\int_{u\sim \frac{\sqrt{Mz}}{c}}\frac{J_{n+2ir_1}(u)}{u^{n-1-2ir_1}}u^{2n-1}h_n(u)du,$$

where

$$h_0(u) = f(u),$$

$$h_1(u) = f'(u),$$

$$h_n(u) = (u^{-1}h_{n-1}(u))' \text{ for } n \geqslant 2.$$

By induction for $n \geqslant 1$

$$u^{2n-1}h_n(u) = \sum_{i=0}^{n} c(i,n)f^{(i)}(u)u^i$$

with

$$f^{(i)}(u)u^i \ll \sum_{j+l+m=i}(g_1^{(j)}(u)u^j)(g_2^{(l)}(u)u^l)u^{2ir_1} \ll \sum_{j+m<i}(g_1^{(j)}(u)u^j)(g_2^{(i-j-m)}(u)u^{i-j-m}).$$

Faá di Bruno's formula (D.2) and the estimate (C.7) give

$$u^j g_1^{(j)}(u) = u^j\frac{\partial^j}{\partial u^j}\left(J_{k-1}\left(u\sqrt{\frac{y}{z}}\right)\right) = \left(\sqrt{\frac{y}{z}}u\right)^j J_{k-1}^{(j)}\left(u\sqrt{\frac{y}{z}}\right) \ll \frac{\left(u\sqrt{\frac{y}{z}}\right)^{k-1}(1+u\sqrt{\frac{y}{z}})^j}{(1+u\sqrt{\frac{y}{z}})^{k-1/2}}.$$

Applying (D.2) to the second function, we obtain

$$u^{i-j-m}g_2^{(i-j-m)}(u) = u^{i-j-m}\frac{\partial^{i-j-m}}{\partial u^{i-j-m}}F_{M,N}\left(\frac{c^2u^2}{16\pi^2 z},y\right)$$

$$= u^{i-j-m}\sum_{\substack{(m_1,m_2)\\ m_1+2m_2=i-j-m}}\frac{(i-j-m)!}{m_1!m_2!(2!)^{m_2}}F_{M,N}^{(m_1+m_2)}\left(\frac{c^2u^2}{16\pi^2 z},y\right)\left(\frac{2c^2}{16\pi^2 z}u\right)^{m_1}\left(\frac{2c^2}{16\pi^2 z}\right)^{m_2}.$$

Then the bound (4.27) implies

$$u^{i-j-m}g_2^{(i-j-m)}(u) \ll \sum_{\substack{(m_1,m_2)\\ m_1+2m_2=i-j-m}} (\frac{c^2}{16\pi^2 z}u^2)^{m_1+m_2} F_{M,N}^{(m_1+m_2)}(\frac{c^2u^2}{16\pi^2 z}, y)$$

$$\ll_{t_1,t_2} P(r_1)P(r_2)(MN)^{-1/2}q^{|t_1|+|t_2|}.$$

The $J$ Bessel function can be bounded trivially

$$J_{n+2ir_1}(u) \ll 1.$$

Then

$$G_1(z,y) \ll_{t_1,t_2} P(r_1)P(r_2)q^{|t_1|+|t_2|}\left(\frac{Qc}{\sqrt{Mz}}\right)^n \frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2}$$

for every integer $n > 0$. The same estimate is valid for $G_{r_1}^-(z,y)$. So, if $z > Z$, the value of $G_{r_1}^-(z,y)$ is small.

Suppose $z \leqslant Z$, then we estimate $G_{r_1}^-(z,y)$ directly (without integration by parts)

$$G_{r_1}^-(z,y) \ll_{t_1,t_2} P(r_1)P(r_2)q^{|t_1|+|t_2|}\frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2}.$$

Let $Y := N$. Since $y \in [N/2, 3N]$, we can add a multiple $(1+\frac{y}{Y})^{-n_2}$.

Combining two estimates for $G_{r_1}^-(z,y)$ in one, we have that for all positive $n_1$ and $n_2$

$$G_{r_1}^-(z,y) \ll_{t_1,t_2} (1+\frac{z}{Z})^{-n_1}(1+\frac{y}{Y})^{-n_2}P(r_1)P(r_2)q^{|t_1|+|t_2|}\frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2}.$$

Analogously, using relation (C.6) and bound (C.9) for $K$-Bessel function, we estimate $G_{r_1}^+(z,y)$.

Finally, differentiating $G_{r_1}^{\pm}(z,y)$ in $z$ variable $j$ times and in $y$ variable $i$ times, we find

$$z^j y^i \frac{\partial^j}{\partial z^j}\frac{\partial^i}{\partial y^i}G_{r_1}^{\pm}(z,y) \ll_{t_1,t_2}$$

$$P(r_1)P(r_2)q^{|t_1|+|t_2|}(1+\frac{z}{Z})^{-n_1}(1+\frac{y}{Y})^{-n_2}\frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{j+i-k+1/2}.$$

for all positive $n_1$ and $n_2$. An extra multiple of $Q^{i+j}$ is obtained by differentiating Bessel functions under the integral. Indeed, by Faá di Bruno's formula (D.2),

$$z^j \frac{\partial^j}{\partial z^j} J_{2ir_1}(\alpha\sqrt{z}) \quad = \quad z^j \sum \binom{j}{m_1, m_2, \ldots, m_j} J_{2ir_1}^{(m_1+m_2+\ldots+m_j)}(\alpha\sqrt{z}) \ \cdot \ \prod_{n=1}^{j} \left(\frac{\alpha z^{1/2-n}}{n!}\right)^{m_n},$$

where

$$\alpha := \frac{4\pi\sqrt{x}}{c}$$

and the sum is over all $j$-tuples $(m_1, m_2, \ldots, m_j)$ such that $1 \cdot m_1 + 2 \cdot m_2 + \ldots j \cdot m_j = j$. Formula (C.13) gives

$$J_{2ir_1}^{(b)}(z) = \frac{1}{2^b} \sum_{t=0}^{b} (-1)^t \binom{b}{t} J_{2ir_1-b+2t}(z).$$

When $z > Z$, the maximum of $z^j \frac{\partial^j}{\partial z^j} J_{2ir_1}(\alpha\sqrt{z})$ is attained when $m_1 + m_2 + \ldots + m_j = j$. Therefore,

$$z^j \frac{\partial^j}{\partial z^j} J_{2ir_1}(\alpha\sqrt{z}) \ll (\alpha\sqrt{z})^j \sum_{t=0}^{j} J_{2ir_1-j+2t}(\alpha\sqrt{z}).$$

This gives an extra multiple $\left(\frac{\sqrt{Mz}}{c}\right)^j$ and

$$G_{r_1}^{-}(z,y) \ll_{t_1,t_2} P(r_1)P(r_2)q^{|t_1|+|t_2|} \left(\frac{Qc}{\sqrt{Mz}}\right)^{n-j} Q^j \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2}$$

for every integer $n > 0$.

In the similar manner

$$y^i \frac{\partial^i}{\partial y^i} \left[J_{k-1}\left(\alpha\sqrt{y}\right) F_{M,N}(x,y)\right] \ll \sum_{a=0}^{i} y^a \left(J_{k-1}(\alpha\sqrt{y})\right)^{(a)} y^{i-a} F_{M,N}(x,y)^{(i-a)}$$

gives an extra factor of $Q^i$. $\square$

## 5.3 Applying Theorem 5.1.2

According to the formula (4.52), the off-off-diagonal term is equal to

$$M^{OOD} = M^{OOD}(0) - \tau_{1/2+ir_2}(\rho)M^{OOD}(1) + M^{OOD}(2), \tag{5.28}$$

where for $B = 0, 1, 2$

$$M^{OOD}(B) = 2\pi i^{-k}\hat{q}^{-2t_1-2t_2}\Bigg(\sum_{q|c}\frac{1}{c^2}\sum_{h\neq 0}S(0,h,c)(T_h^-(c,B)+T_h^+(c,B))$$
$$-\frac{1}{\rho}\sum_{\frac{q}{\rho}|c}\frac{1}{c^2}\sum_{h\neq 0}S(0,h,c)(T_h^-(c,B)+T_h^+(c,B))\Bigg).$$

Since $k$ is even, $i^{-k} = i^k$.

**Lemma 5.3.1.** *Up to an error term*

$$O_{\epsilon,\rho,t_1,t_2}(P(r_1)P(r_2)q^{\epsilon}q^{|t_1|-t_1+|t_2|-t_2}(q^{-\frac{2k-3}{12}} + q^{-1/4})),$$

*we have*

$$T_h^{\mp}(c,B) = \pm\int_0^{\infty}\delta_{h\pm\rho^B y>0}G_{r_1}^{\mp}(h\pm\rho^B y,\rho^B y)\Lambda(h\pm\rho^B y,\rho^B y)dy \tag{5.29}$$

*with*

$$\Lambda(h\pm\rho^B y,\rho^B y) := \sum_{w=1}^{\infty}S(0,h,w)\sum_{\epsilon_1,\epsilon_2=\pm 1}\frac{(\rho^B,w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}$$
$$\times\zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)(h\pm\rho^B y)^{i\epsilon_1 r_1}y^{i\epsilon_2 r_2}. \tag{5.30}$$

*Proof.* We apply Theorem 5.1.2 to the function $T_h^{\mp}(c,B)$ and let $x = h\pm\rho^B y$. Then

$$T_h^{\mp}(c,B) = \pm\int_0^{\infty}\delta_{h\pm\rho^B y>0}G_{r_1}^{\mp}(h\pm\rho^B y,\rho^B y)\Lambda(h\pm\rho^B y,\rho^B y)dy + O(ET),$$

where

$$\Lambda(h \pm \rho^B y, \rho^B y) := \sum_{w=1}^{\infty} S(0,h,w) \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{(\rho^B, w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \times \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)(h \pm \rho^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}$$

and the error term is

$$ET := P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2} \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2} Q^{5/4}(Z+N)^{1/4}(ZN)^{1/4+\epsilon}.$$

Since $Z = Q^2 \frac{c^2}{M} > N$,

$$ET \ll P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2} M^{1/2} N^{1/4} \left(\frac{\sqrt{MN}}{c}\right)^{k-2} Q^{-k+11/4}.$$

Note that $T_h^{\mp}(c)$ is small when $|h| \gg Zq^{\epsilon}$ because $G^{\mp}$ is small when $z \gg Zq^{\epsilon}$. This allows adding $\left(1+\frac{|h|}{Z}\right)^{-2}$ into the error term $ET$. Multiplying by $S(0,h,c)$ and summing over $h$, we have

$$ET_1 := \sum_h S(0,h,c)\left(1+\frac{|h|}{Z}\right)^{-2} ET \ll P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2} c^2 N^{1/4} M^{-1/2} \left(\frac{\sqrt{MN}}{c}\right)^{k-2} Q^{-k+2+11/4}.$$

Finally, we sum over $c$. If $k=2$,

$$\sum_{\substack{c \leqslant C \\ q|c}} c^{-2} ET_1 \ll P(r_1)P(r_2)q^{\epsilon} q^{|t_1|-t_1+|t_2|-t_2} \frac{N^{1/4}}{M^{1/2}} \sum_{\substack{c \leqslant C \\ q|c}} \left[1+\left(\frac{\sqrt{MN}}{c}\right)^{11/4}\right] \ll P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2} \left(\frac{N^{1/4}}{M^{1/2}}\frac{C}{q} + \frac{N^{13/8}M^{7/8}}{q^{11/4}}\right).$$

An optimal value of $C$ can be found by making equal the first summand and the error term in Lemma 4.4.4

$$\frac{N^{1/4}}{M^{1/2}}\frac{C}{q}=\left(\frac{\sqrt{MN}}{C}\right)^{1/2}.$$

Thus, $C:=\min(q^{2/3}M^{1/2},q^{7/6})$ and

$$\sum_{q|c}c^{-2}ET_1\ll P(r_1)P(r_2)q^{\epsilon}q^{|t_1|-t_1+|t_2|-t_2}(q^{-1/12}+q^{-1/4}).$$

If $k\geqslant 4$, then

$$\begin{aligned}\sum_{\substack{c\leqslant C\\ q|c}}c^{-2}ET_1 &\ll P(r_1)P(r_2)q^{\epsilon}q^{|t_1|-t_1+|t_2|-t_2}\frac{N^{1/4}}{M^{1/2}}\sum_{\substack{c\leqslant C\\ q|c}}\left[\left(\frac{\sqrt{MN}}{c}\right)^{k-2}+\left(\frac{\sqrt{MN}}{c}\right)^{11/4}\right]\\ &\ll P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2}\left(\frac{N^{1/4}}{M^{1/2}}\left(\frac{\sqrt{MN}}{q}\right)^{k-2}+\frac{N^{13/8}M^{7/8}}{q^{11/4}}\right)\\ &\ll P(r_1)P(r_2)q^{|t_1|-t_1+|t_2|-t_2}q^{-1/4}.\end{aligned}$$

Combining two estimates in one, we have that for any even $k$

$$\sum_{\substack{c\leqslant C\\ q|c}}c^{-2}ET_1\ll P(r_1)P(r_2)q^{\epsilon}q^{|t_1|-t_1+|t_2|-t_2}(q^{-\frac{2k-3}{12}}+q^{-1/4}).$$

□

## 5.4 Extension of summations

Analogously to the off-diagonal term, at the cost of admissible error, we can reintroduce summation over $\max(M,N)\geqslant q^{1+\epsilon}$ and extend the summation over $c$ up to some large value $C_{max}=q^{\Omega}$.

**Proposition 5.4.1.** *For $l=0,1$, we have*

$$\sum_{\max(M,N)\ll q^{1+\epsilon}}\sum_{\substack{\frac{q}{\rho^l}|c\\ c>C}}\frac{1}{c^2}\sum_{h\neq 0}S(0,h,c)\sum_{m\pm n\rho^B=h}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)G^{\pm}_{r_1}(m,n\rho^B)\ll_{\epsilon,\rho,t_1,t_2}$$

$$P(r_1)P(r_2)q^{|t_1|+|t_2|+\epsilon}(q^{-\frac{2k-3}{12}}+q^{-1/4}).\quad(5.31)$$

*Proof.* This estimate can be obtained using the large sieve inequality (2.28). More precisely, we repeat the proof of Lemma 4.4.4 with

$$g(m,n,c_1) = \frac{1}{c_1} G^{\pm}_{r_1}(m, n\rho^B)$$

and

$$X = P(r_1)P(r_2)q^{-|t_1|-|t_2|}\frac{\sqrt{N}}{\sqrt{M}}\left(\frac{\sqrt{MN}}{c}\right)^{-k+1}.$$

□

**Proposition 5.4.2.** *For any $\epsilon > 0$, any $A > 0$ and $l = 0,1$*

$$\sum_{\frac{q}{\rho^l}|c} \frac{1}{c^2} \sum_{\max(M,N) \gg q^{1+\epsilon}} \sum_{h \neq 0} S(0,h,c) \sum_{m \pm n\rho^B = h} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) G^{\pm}_{r_1}(m, n\rho^B)$$

$$\ll_{\epsilon,\rho,A,t_1,t_2} P(r_1)P(r_2)q^{-A}. \quad (5.32)$$

*Proof.* This follows from the rapid decay of $F_{M,N}$ when $\max(M,N) \gg q^{1+\epsilon}$. See proof of proposition 4.4.2 for details. □

Now it is possible to combine all functions $F_M$ into $F$ and replace $\sum_{M,N} F_{M,N}$ by

$$F(x,y) := \frac{1}{\sqrt{xy}} W_{t_1,r_1}(\frac{x}{\hat{q}^2}) W_{t_2,r_2}(\frac{y}{\hat{q}^2}) F(x)F(y), \quad (5.33)$$

where $F(x)$ is a smooth function, compactly supported in $[1/2, \infty)$ such that $F(x) = 1$ for $x \geqslant 1$.

## 5.5 Expression for the off-off-diagional term

**Lemma 5.5.1.** *One has*

$$M^{OOD}(B) = 2\pi i^k \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1,\epsilon_2=\pm 1} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)$$

$$\times \left( \sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{q|cg \\ cg<q^\Omega}} \frac{1}{c} \sum_{w} \frac{(\rho^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h\neq 0 \\ [w,c]|h}} \big(V^-(h)+V^+(h)\big) \right.$$

$$\left. -\frac{1}{\rho} \sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{\frac{q}{\rho}|cg \\ cg<q^\Omega}} \frac{1}{c} \sum_{w} \frac{(\rho^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h\neq 0 \\ [w,c]|h}} \big(V^-(h)+V^+(h)\big) \right),$$

*where*

$$V^-(h)+V^+(h) = -\frac{1}{(2\pi i)^2} \frac{1}{\rho^{B(1+i\epsilon_2 r_2)}}$$

$$\times \int_{\Re\beta=0.7} \int_{\Re z=-0.1} \frac{\Gamma(\beta+ir_1)\Gamma(\beta-ir_1)(4\pi)^{k+2z-2\beta}2^{-k-2z+2\beta}}{\Gamma(1+z)\Gamma(k+z)\sin(\pi z)} (cg)^{-k+1-2z+2\beta}$$

$$\times h^{k/2+z-\beta+i\epsilon_1 r_1+i\epsilon_2 r_2} \int_{x=0}^{\infty} x^{z-\beta+k/2} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) F(x) \frac{dx}{x} \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2} W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right) F(hy)$$

$$\times \left( \frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1} \frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1} \frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}} \right) \frac{dy}{y} dz d\beta.$$

*Proof.* Lemma 5.3.1 implies

$$T_h^-(c,B)+T_h^+(c,B) = \sum_{w=1}^{\infty} S(0,h,w) \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{(\rho^B,w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)$$

$$\times \int_0^\infty \big[\delta_{h+\rho^B y>0} G_{r_1}^-(h+\rho^B y, \rho^B y)(h+\rho^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}$$

$$+\delta_{h-\rho^B y>0} G_{r_1}^+(h-\rho^B y, \rho^B y)(h-\rho^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}\big]\, dy.$$

We plug in the expressions for $G_{r_1}^-$ and $G_{r_1}^+$ given by (4.47) and (4.48) and use the identity

$$F(x,\rho^B y) = \frac{1}{(\rho^B xy)^{1/2}} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{\rho^B y}{\hat{q}^2}\right) F(x)F(\rho^B y).$$

This gives

$$T_h^-(c,B) + T_h^+(c,B) = \sum_{w=1}^{\infty} S(0,h,w) \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{(\rho^B, w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)$$

$$\times 2\pi \int_0^\infty \int_0^\infty \frac{1}{(\rho^B xy)^{1/2}} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{\rho^B y}{\hat{q}^2}\right) J_{k-1}\left(\frac{4\pi\sqrt{x\rho^B y}}{c}\right)$$

$$\times \left[\delta_{h+\rho^B y>0} k_0 \left(\frac{4\pi\sqrt{x(h+\rho^B y)}}{c}, 1/2+ir_1\right)(h+\rho^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}\right.$$

$$\left. +\delta_{h-\rho^B y>0} k_1 \left(\frac{4\pi\sqrt{x(h+\rho^B y)}}{c}, 1/2+ir_1\right)(h-\rho^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}\right] F(x)F(\rho^B y)dxdy.$$

The off-off-diagonal term

$$M^{OOD}(B) = 2\pi i^k \hat{q}^{-2t_1-2t_2} \left(\sum_{q|c} \frac{1}{c^2} \sum_{h\neq 0} S(0,h,c)(T_h^-(c,B)+T_h^+(c,B))\right.$$

$$\left. -\frac{1}{\rho} \sum_{\frac{q}{\rho}|c} \frac{1}{c^2} \sum_{h\neq 0} S(0,h,c)(T_h^-(c,B)+T_h^+(c,B))\right)$$

contains two Ramanujan sums $S(0,h,c)$ and $S(0,h,w)$. Applying the formulas

$$S(0,h,c) = \sum_{\substack{gc_1=c\\ c_1|h}} \mu(g)c_1,$$

$$S(0,h,w) = \sum_{\substack{vw_1=c\\ w_1|h}} \mu(v)w_1,$$

we obtain

$$M^{OOD}(B) = 2\pi i^k \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1,\epsilon_2=\pm 1} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)$$

$$\times \left( \sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{q|cg \\ cg<q^{\Omega}}} \frac{1}{c} \sum_{w} \frac{(\rho^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h\neq 0 \\ [w,c]|h}} \big(V^-(h)+V^+(h)\big)\right.$$

$$\left. -\frac{1}{\rho} \sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{\frac{q}{\rho}|cg \\ cg<q^{\Omega}}} \frac{1}{c} \sum_{w} \frac{(\rho^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h\neq 0 \\ [w,c]|h}} \big(V^-(h)+V^+(h)\big)\right),$$

where

$$V^-(h)+V^+(h) = 2\pi \int_0^{\infty}\int_0^{\infty} \frac{1}{(\rho^B xy)^{1/2}} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{\rho^B y}{\hat{q}^2}\right) J_{k-1}\left(\frac{4\pi\sqrt{x\rho^B y}}{cg}\right)$$

$$\times \left[ \delta_{h+\rho^B y>0} k_0 \left(\frac{4\pi\sqrt{x(h+\rho^B y)}}{cg}, 1/2+ir_1\right)(h+\rho^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}\right.$$

$$\left. +\delta_{h-\rho^B y>0} k_1 \left(\frac{4\pi\sqrt{x(h+\rho^B y)}}{cg}, 1/2+ir_1\right)(h-\rho^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}\right] F(x)F(\rho^B y)\,dx\,dy.$$

In the expression $V^-(h)+V^+(h)$ we replace negative $h$ by their absolute value and make a change of variables $\frac{\rho^B y}{h} \to y$ in the integral. As a result,

$$V^-(h) + V^+(h) = 2\pi \frac{h^{1/2+i\epsilon_1 r_1 + i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}}$$
$$\times \int_0^\infty \int_0^\infty \frac{y^{i\epsilon_2 r_2}}{(xy)^{1/2}} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right) J_{k-1}\left(\frac{4\pi\sqrt{xhy}}{cg}\right)$$
$$\times \left[ k_0\left(\frac{4\pi\sqrt{xh(1+y)}}{cg}, 1/2 + ir_1\right)(1+y)^{i\epsilon_1 r_1}\right.$$
$$+ \delta_{y>1} k_0\left(\frac{4\pi\sqrt{xh(y-1)}}{cg}, 1/2 + ir_1\right)(-1+y)^{i\epsilon_1 r_1}$$
$$\left. + \delta_{y<1} k_1\left(\frac{4\pi\sqrt{xh(1-y)}}{cg}, 1/2 + ir_1\right)(1-y)^{i\epsilon_1 r_1}\right] F(x)F(hy)dxdy.$$

Finally, we apply Mellin transforms of Bessel functions (E.5), (E.11) and (E.12)

$$V^-(h) + V^+(h) = -\frac{1}{(2\pi i)^2}\frac{1}{\rho^{B(1+i\epsilon_2 r_2)}}$$
$$\times \int_{\Re\beta=0.7}\int_{\Re z=-0.1} \frac{\Gamma(\beta+ir_1)\Gamma(\beta-ir_1)(4\pi)^{k+2z-2\beta}2^{-k-2z+2\beta}}{\Gamma(1+z)\Gamma(k+z)\sin(\pi z)}(cg)^{-k+1-2z+2\beta}$$
$$\times h^{k/2+z-\beta+i\epsilon_1 r_1+i\epsilon_2 r_2}\int_{x=0}^\infty x^{z-\beta+k/2} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) F(x)\frac{dx}{x}\int_{y=0}^\infty y^{z+k/2+i\epsilon_2 r_2} W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right) F(hy)$$
$$\times \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1}\frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1}\frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}}\right)\frac{dy}{y}dzd\beta.$$

Note that the contour of integration $(*)$ (see figure E.1) is shifted to $\Re\beta = 0.7$, which is possible due to the rapid decay of the $x$ integral in $\beta$. The change of the order of integration in $V^-(h) + V^+(h)$ is justified by absolute convergence of all integrals. □

## 5.6 Replacing $F(x)F(hy)$ by 1 on the interval $[0, \infty)^2$

This step allows us to simplify the integration and can be performed with a cost of negligible error.

### 5.6.1 y-integral

Consider

$$IY := \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2} W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right) F(hy)$$

$$\times \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1}\frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1}\frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}}\right)\frac{dy}{y}. \quad (5.34)$$

**Lemma 5.6.1.** *The function $F(hy)$ can be replaced by 1 in $IY$ with an error*

$$O_{\epsilon,\rho}(P(r_1)P(r_2)q^{-t_1-t_2}q^{-1/2+\epsilon}).$$

*Proof.* $F(hy)$ is a smooth function, compactly supported in $[1/2,\infty)$ such that $F(hy)=1$ for $hy \geqslant 1$. Thus, we only need to estimate the integral for $y < 1/h$ . It is bounded by $\left(\frac{1}{h}\right)^{k/2+\Re z}\cos\pi\beta$. We are left to estimate

$$T := \sum_{g,v,w}\frac{1}{g^2v^2w}\sum_{\substack{q|cg\\ cg<q^{\Omega}}}\frac{1}{c}\sum_{[c,w]|h} h^{-\beta}\int_{\Re\beta=0.7}\int_{\Re z=-0.1}\frac{\Gamma(\beta+ir_1)\Gamma(\beta-ir_1)\cos\pi\beta}{\Gamma(1+z)\Gamma(k+z)\sin(\pi z)}$$

$$\times (cg)^{-k+1-2z+2\beta}\int_{x=0}^{\infty} x^{z-\beta+k/2}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)F(x)\frac{dx}{x}.$$

To make the sums over $h$ and $w$ absolutely convergent, one has to move $\beta$ contour to the right $\Re\beta > 1$. At the same time, partial integration shows that the $x$-integral decays rapidly in $\beta$:

$$\int_0^{\infty} x^{z-\beta+k/2}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)F(x)\frac{dx}{x} = \frac{1}{(z-\beta+k/2)(z-\beta+k/2)\dots(z-\beta+k/2+n-1)}$$

$$\times\int_0^{\infty}\frac{\partial^n}{\partial x^n}\left(W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)F(x)\right)x^{z-\beta+k/2+n-1}dx \ll P(r_1)P(r_2)\frac{1}{|\beta|^n}q^{z-\beta+k/2}.$$

Assume that $\Re\beta > 1$. We have

$$T \ll P(r_1)P(r_2)q^{z-\beta+k/2} \sum_{v,w,h} \frac{1}{v^2 w^{1+\beta} h^\beta} \sum_{\substack{q|cg \\ cg<q^\Omega}} \frac{1}{c^{1+\beta} g^2} (cg)^{-k+1-2z+2\beta}$$

$$\ll P(r_1)P(r_2)q^{z-\beta+k/2} \sum_{\substack{q|cg \\ cg<q^\Omega}} (cg)^{-k-1-2z+2\beta} \ll P(r_1)P(r_2)q^{z-\beta+k/2-1} q^{\Omega(-k-2z+2\beta)}.$$

Moving $\beta$ contour to $\Re\beta = k/2 + \delta$ and $z$ contour to $-\delta$, $M^{OOD}$ is dominated by

$$P(r_1)P(r_2)q^{-t_1-t_2}q^{-1}q^{4\delta\Omega-2\delta}.$$

Choosing $\delta = \frac{1}{4(2\Omega+1)}$, we obtain the result. $\square$

**Lemma 5.6.2.** *One has*

$$IY = \frac{1}{2\pi i} \int_{\Re t = k/2-0.2} \frac{G(t)}{G(t_2)} \frac{\Gamma(t+ir_2+k/2)\Gamma(t-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)} \zeta_q(1+2t) \left(\frac{\hat{q}^2}{h}\right)^t$$
$$\times \frac{\Gamma(k/2+z-t+i\epsilon_2 r_2)\Gamma(-k/2-z+t+\beta-i\epsilon_1 r_1 - i\epsilon_2 r_2)}{\Gamma(\beta - i\epsilon_1 r_1)}$$
$$\times \left(\cos(\pi\beta) + \frac{\cos(\pi\beta)\sin(\pi(k/2+z-t+i\epsilon_2 r_2))}{\sin(\pi(\beta - i\epsilon_1 r_1))}\right.$$
$$\left. + \frac{\cos(\pi i r_1)\sin(\pi(-k/2-z+t+\beta-i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(\beta-i\epsilon_1 r_1))}\right) \frac{2t dt}{t^2 - t_2^2}.$$

*Proof.* By Lemma 5.6.1, the $y$-integral is equal to

$$IY = \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2} W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right)$$
$$\times \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1}\frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1}\frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}}\right)\frac{dy}{y}.$$

We plug in the expression

$$W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right) = \frac{1}{2\pi i}\int_{\Re t=k/2-0.2} \frac{G(t)}{G(t_2)} \zeta_q(1+2t) \frac{\Gamma(t+ir_2+k/2)\Gamma(t-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}$$
$$\times \left(\frac{hy}{\hat{q}^2}\right)^{-t} \frac{2t dt}{t^2-t_2^2}.$$

Note that we shifted $\Re t$ from 3 to $k/2-0.2$ without crossing any pole[1]. Then

$$IY = \frac{1}{2\pi i}\int_{\Re t=k/2-0.2}\frac{G(t)}{G(t_2)}\frac{\Gamma(t+ir_2+k/2)\Gamma(t-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}\zeta_q(1+2t)\left(\frac{\hat{q}^2}{h}\right)^t$$

$$\times\int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2-t}\left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}}+\delta_{y>1}\frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}}+\delta_{y<1}\frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}}\right)\frac{dy}{y}\frac{2tdt}{t^2-t_2^2}.$$

Mellin transforms (E.0.26), (E.0.27), (E.0.28), and Euler's reflection formula (A.9) give

$$IY = \frac{1}{2\pi i}\int_{\Re t=k/2-0.2}\frac{G(t)}{G(t_2)}\frac{\Gamma(t+ir_2+k/2)\Gamma(t-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}\zeta_q(1+2t)\left(\frac{\hat{q}^2}{h}\right)^t$$

$$\times\left(\cos(\pi\beta)+\frac{\cos(\pi\beta)\sin(\pi(k/2+z-t+i\epsilon_2 r_2))}{\sin(\pi(\beta-i\epsilon_1 r_1))}\right.$$

$$\left.+\frac{\cos(\pi i r_1)\sin(\pi(-k/2-z+t+\beta-i\epsilon_1 r_1-i\epsilon_2 r_2))}{\sin(\pi(\beta-i\epsilon_1 r_1))}\right)$$

$$\times\frac{\Gamma(k/2+z-t+i\epsilon_2 r_2)\Gamma(-k/2-z+t+\beta-i\epsilon_1 r_1-i\epsilon_2 r_2)}{\Gamma(\beta-i\epsilon_1 r_1)}\frac{2tdt}{t^2-t_2^2}.$$

□

**Remark 5.6.3.** *If*

$$\beta-i\epsilon_1 r_1 = 0,-1,-2,\ldots,$$

*poles of* $1/\sin(\pi(\beta-i\epsilon_1 r_1))$ *in* $IY$ *are cancelled by zeroes of* $1/\Gamma(\beta-i\epsilon_1 r_1)$*. Poles at*

$$\beta-i\epsilon_1 r_1 = j \text{ with } j=1,2,3\ldots$$

*are compensated by vanishing numerator.*

### *5.6.2 x-integral*

**Lemma 5.6.4.** *The function* $F(x)$ *can be replaced by* 1 *in the expression* $V^-(h)+V^+(h)$ *at the cost of negligible error*

$$P(r_1)P(r_2)q^{-t_1-t_2+\epsilon}q^{-k/2+0.5}.$$

[1] This step is required to ensure that all poles of $\Gamma(-k/2-z+t+\beta-i\epsilon_1 r_1-i\epsilon_2 r_2)$ lie to the left of the $t$ contour

*Proof.* We show that the contribution of $F_1(x) = 1 - F(x)$ is negligible. Note that $F_1(x) = 0$ for $x \geqslant 1$ since in that case $F(x) = 1$. The part of $M^{OOD}$, which affects the $x$-integral, can be written as follows

$$\sum_{v,w} \frac{1}{v^2 w} \sum_{\substack{c,g \\ q|cg}} \sum_{[c,w]|h} g^{-k-1-2z+2\beta} c^{-k-2z+2\beta} h^{k/2+z-\beta-t} q^t \Gamma(-k/2 - z + t + \beta - i\epsilon_1 r_1 - i\epsilon_2 r_2)$$

$$\times \Gamma(k/2 + z - t + i\epsilon_2 r_2) H_1(t, z, \beta) \int_0^1 x^{z-\beta+k/2} W_{t_1,r_1}(\frac{x}{\hat{q}^2}) F_1(x) \frac{dx}{x}.$$

Here $H_1$ is an analytic function. We have

$$\Re z = -0.1,$$

$$\Re\beta = 0.7,$$

$$\Re t = k/2 - 0.2.$$

Without crossing any pole, we shift $\beta$-contour to

$$\Re\beta = 0.3.$$

In order to make the sums over $h$ and $w$ absolutely convergent, we move $t$ contour to

$$\Re t = k/2 + 0.7,$$

crossing a pole at $t = k/2 + z + i\epsilon_2 r_2$. Since $\Re z - \Re\beta + k/2 > 0$, the $x$-integral can be integrated by parts $n$ times (for sufficiently large $n$) to make $\beta$-integral convergent. This gives

$$\int_0^1 x^{z-\beta+k/2} W_{t_1,r_1}(\frac{x}{\hat{q}^2}) F_1(x) \frac{dx}{x} \ll P(r_1) P(r_2) \frac{1}{|\beta|^n}.$$

Finally, all sums and integrals are absolutely convergent and $q^{-k-1+t-2z+2\beta}$ can be factored out due to divisibility conditions. In total, this gives an error

$$P(r_1) P(r_2) q^{-t_1-t_2+\epsilon} q^{-k/2+0.5}.$$

For the pole at $t = k/2+z+i\epsilon_2 r_2$ another contour shift is required to make all sums absolutely convergent. We move $z$-contour to

$$\Re z = 0.5 + 2\epsilon$$

and $\beta$-contour to

$$\Re\beta = 1 + \epsilon.$$

Note that the pole of $1/\sin(\pi z)$ at $z = 0$ is cancelled by the zero of $G(t) = G(k/2+z+i\epsilon_2 r_2)$. The $x$ integral is bounded by $P(r_1)P(r_2)\frac{1}{|\beta|^n}$. The power of $q$, corresponding to divisibility conditions on $g, c, h$, is $q^{-k-1+t-2z+2\beta}$. This gives an error term

$$P(r_1)P(r_2)q^{-t_1-t_2}q^{-k/2+0.5}.$$

□

**Proposition 5.6.5.** *One has*

$$M^{OOD}(B) = 2\pi i^k \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1,\epsilon_2=\pm1} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)$$

$$\times\left(\sum_{g,v}\frac{\mu(g)}{g^2}\frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\sum_{\substack{q|cg\\ cg<q^{\Omega}}}\frac{1}{c}\sum_{w}\frac{(\rho^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\sum_{\substack{h\neq 0\\ [w,c]|h}}\big(V^-(h)+V^+(h)\big)\right.$$

$$\left.-\frac{1}{\rho}\sum_{g,v}\frac{\mu(g)}{g^2}\frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\sum_{\substack{\frac{q}{\rho}|cg\\ cg<q^{\Omega}}}\frac{1}{c}\sum_{w}\frac{(\rho^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\sum_{\substack{h\neq 0\\ [w,c]|h}}\big(V^-(h)+V^+(h)\big)\right),$$

*where*

$$V^-(h)+V^+(h) = -\frac{i^k}{(2\pi i)^3}\int_{\Re t=k/2+0.7}\int_{\Re s=k/2-0.4}\int_{\Re z=-0.1}\frac{\hat{q}^{2s+2t}}{\rho^{B(1+i\epsilon_2 r_2)}}(cg)^{1-2s}h^{s-t+i\epsilon_1 r_1+i\epsilon_2 r_2}$$

$$\times (2\pi)^{2s}\frac{G(s)G(t)}{G(t_1)G(t_2)}\zeta_q(1+2t)\zeta_q(1+2s)\frac{\Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)}{\sin(\pi z)\Gamma(1+z)\Gamma(k+z)}$$

$$\times\frac{\Gamma(k/2+s\pm ir_1)\Gamma(k/2+t\pm ir_2)\Gamma(k/2+z-s+i\epsilon_1 r_1)\Gamma(k/2+z-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_1\pm ir_1)\Gamma(k/2+t_2\pm ir_2)}$$

$$\times\left(\cos(\pi(z-s))+\frac{\cos(\pi(z-s))\sin(\pi(z-t+i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))}\right.$$

$$\left.+\frac{\cos(\pi i r_1)\sin(\pi(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))}\right)dz\frac{2sds}{s^2-t_1^2}\frac{2tdt}{t^2-t_2^2}$$

*plus the contribution of poles at $t=k/2+z+i\epsilon_2 r_2$ ( while shifting the $t$-contour to the right).*

**Remark 5.6.6.** *We do not compute the contribution of poles at $t=k/2+z+i\epsilon_2 r_2$ since it will be cancelled by another contour shift in 5.7.1.*

*Proof.* By inverse Mellin transform we have

$$\int_0^\infty x^{z-1-\beta+k/2}W_{t_1,r_1}(\frac{x}{\hat{q}^2})dx = 2\frac{G(z-\beta+k/2)}{G(t_1)}\zeta_q(1+k+2z-2\beta)\hat{q}^{k+2z-2\beta}$$

$$\times\frac{\Gamma(k+z-\beta+ir_1)\Gamma(k+z-\beta-ir_1)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)}\frac{k/2+z-\beta}{(k/2+z-\beta)^2-t_1^2}$$

for $\Re(z-\beta+k/2)>-1$. Then the result follows by letting $s:=k/2+z-\beta$. □

## 5.7 Shifting the $z$-contour

The $z$-integral is given by

$$IZ := \frac{1}{2\pi i}\int_{\Re z=-0.1}\frac{\Gamma(k/2+z-s+i\epsilon_1 r_1)\Gamma(k/2+z-t+i\epsilon_2 r_2)}{\sin(\pi z)\Gamma(1+z)\Gamma(k+z)}$$

$$\times\left(\cos(\pi(z-s))+\frac{\cos(\pi(z-s))\sin(\pi(z-t+i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))}\right.$$

$$\left.+\frac{\cos(\pi i r_1)\sin(\pi(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))}\right)dz. \quad (5.35)$$

Stirling's formula implies that the integrand decays as $|z|^{-1-s-t}$. We shift $\Re z$ to $D > 0$ and then let $D \to +\infty$. This leads to three types of possible poles described in the table below.

| Possible poles at | Coming from function |
|---|---|
| $z = t - k/2 - i\epsilon_2 r_2$ | $\Gamma(k/2 + z - t + i\epsilon_2 r_2)$ |
| $z = n + s + i\epsilon_1 r_1$ | $1/\sin(\pi(z - s - i\epsilon_1 r_1))$ |
| $z = n, n \geqslant 0$ | $1/\sin(\pi z)$ |

### 5.7.1 *Poles at* $z = t - k/2 - i\epsilon_2 r_2$

Residues at these poles cancel whose mentioned in remark 5.6.6 (while performing the shift of $t$ to the right). Consider

$$\int_t \int_z \Gamma(k/2 + z - t + i\epsilon_2 r_2) f(z,t) dz dt.$$

Shifting $t$ integral to the right, we have the residue

$$-Res_{z=-k/2+t-i\epsilon_2 r_2} \Gamma(k/2 + z - t + i\epsilon_2 r_2) f(z,t).$$

Moving $z$ to the right, we obtain

$$-Res_{t=k/2+z+i\epsilon_2 r_2} \Gamma(k/2 + z - t + i\epsilon_2 r_2) f(z,t).$$

Since $z$ and $t$ have different signs in $\Gamma(k/2 + z - t + i\epsilon_2 r_2)$, these residues cancel each other.

### 5.7.2 *Poles at* $z = n + s + i\epsilon_1 r_1$

**Proposition 5.7.1.** *The final expression is holomorphic at* $z = n + s + i\epsilon_1 r_1$.

*Proof.* To show this, we write

$$\sin(\pi(z - t + i\epsilon_2 r_2)) = -\sin(\pi(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2)) \cos(\pi(z - s - i\epsilon_1 r_1)) + \cos(\pi(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2)) \sin(\pi(z - s - i\epsilon_1 r_1))$$

and plug it in $IZ$. After simplifications,

$$IZ := \frac{1}{2\pi i} \int_{\Re z = -0.1} \frac{\Gamma(k/2 + z - s + i\epsilon_1 r_1)\Gamma(k/2 + z - t + i\epsilon_2 r_2)}{\sin(\pi z)\Gamma(1 + z)\Gamma(k + z)} [\cos(\pi(z - s)) + \cos(\pi(z - s)) \cos(\pi(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2)) + \sin(\pi(z - s)) \sin(\pi(z - s + i\epsilon_1 r_1))].$$

This is holomorphic at $z = n + s + i\epsilon_1 r_1$. $\square$

### 5.7.3 Poles at $z = n, n \geqslant 0$

**Proposition 5.7.2.** *The poles at $z = n$ are simple and its contribution is given by*

$$-\frac{1}{\pi}\Gamma(s+t-i\epsilon_1 r_1 - i\epsilon_2 r_2)\frac{\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+s-i\epsilon_1 r_1)\Gamma(k/2+t-i\epsilon_2 r_2)}$$
$$\times\left[\cos(\pi s)+\cos(\pi(t-i\epsilon_1 r_1-i\epsilon_2 r_2))\right] ds.$$

*Proof.* We need to compute

$$P_1 := -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{\cos(\pi n)}\frac{\Gamma(k/2+n-s+i\epsilon_1 r_1)\Gamma(k/2+n-t+i\epsilon_2 r_2)}{\Gamma(1+n)\Gamma(k+n)}$$
$$\times\left(\cos(\pi(n-s))+\frac{\cos(\pi(n-s))\sin(\pi(n-t+i\epsilon_2 r_2))}{\sin(\pi(n-s-i\epsilon_1 r_1))}\right.$$
$$\left.+\frac{\cos(\pi i r_1)\sin(\pi(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2))}{\sin(\pi(n-s-i\epsilon_1 r_1))}\right).$$

Since $n \in \mathbb{Z}$, we have

$$P_1 := -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma(k/2+n-s+i\epsilon_1 r_1)\Gamma(k/2+n-t+i\epsilon_2 r_2)}{\Gamma(1+n)\Gamma(k+n)}$$
$$\times\left(\cos(\pi s)+\frac{\cos(\pi s)\sin(\pi(t-i\epsilon_2 r_2))}{\sin(\pi(s+i\epsilon_1 r_1))}\right.$$
$$\left.-\frac{\cos(\pi i r_1)\sin(\pi(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2))}{\sin(\pi(s+i\epsilon_1 r_1))}\right).$$

Using Gauss hypergeometric identity D.0.25,

$$\sum_{n=0}^{\infty}\frac{\Gamma(k/2+n-s+i\epsilon_1 r_1)\Gamma(k/2+n-t+i\epsilon_2 r_2)}{\Gamma(1+n)\Gamma(k+n)}$$
$$=\Gamma(s+t-i\epsilon_1 r_1-i\epsilon_2 r_2)\frac{\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+s-i\epsilon_1 r_1)\Gamma(k/2+t-i\epsilon_2 r_2)}.$$

Simplifying the trigonometric part, we obtain

$$\cos(\pi s) + \frac{\cos(\pi s)\sin(\pi(t - i\epsilon_2 r_2))}{\sin(\pi(s + i\epsilon_1 r_1))} - \frac{\cos(\pi i r_1)\sin(\pi(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(s + i\epsilon_1 r_1))}$$
$$= \cos(\pi s) + \cos(\pi(t - i\epsilon_1 r_1 - i\epsilon_2 r_2)).$$

This implies

$$P_1 = -\frac{1}{\pi}\Gamma(s + t - i\epsilon_1 r_1 - i\epsilon_2 r_2)\frac{\Gamma(k/2 - s + i\epsilon_1 r_1)\Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + s - i\epsilon_1 r_1)\Gamma(k/2 + t - i\epsilon_2 r_2)}$$
$$\times \left[\cos(\pi s) + \cos(\pi(t - i\epsilon_1 r_1 - i\epsilon_2 r_2))\right].$$

□

As a result, the off-off-diagonal can be written as follows.

**Proposition 5.7.3.**

$$M^{OOD}(B) = \frac{2}{(2\pi i)^2}\hat{q}^{-2t_1-2t_2}\sum_{\epsilon_1,\epsilon_2=\pm 1}\zeta(1 + 2i\epsilon_1 r_1)\zeta(1 + 2i\epsilon_2 r_2)$$
$$\times \int_{\Re t = k/2+0.7}\int_{\Re s = k/2-0.4}\frac{G(s)G(t)}{G(t_1)G(t_2)}\zeta_q(1 + 2s)\zeta_q(1 + 2t)\frac{q^{s+t}}{(2\pi)^{2t}}$$
$$\times \Gamma(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2)\Gamma(t + s - i\epsilon_1 r_1 - i\epsilon_2 r_2)$$
$$\times \frac{\Gamma(k/2 + s + i\epsilon_1 r_1)\Gamma(k/2 + t + i\epsilon_2 r_2)\Gamma(k/2 - s + i\epsilon_1 r_1)\Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + t_1 + ir_1)\Gamma(k/2 + t_1 - ir_1)\Gamma(k/2 + t_2 + ir_2)\Gamma(k/2 + t_2 - ir_2)}$$
$$\times \left(\sum_{q|cg}\sum_{g}\frac{\mu(g)}{g^{2s+1}}TD(c) - 1/\rho\sum_{q|c\rho g}\sum_{g}\frac{\mu(g)}{g^{2s+1}}TD(c)\right)$$
$$\times \left[\cos(\pi s) + \cos(\pi(t - i\epsilon_1 r_1 - i\epsilon_2 r_2))\right]\frac{2sds}{s^2 - t_1^2}\frac{2tdt}{t^2 - t_2^2},$$

*where*

$$TD(c) = \frac{1}{c^{2s}}\sum_{v}\frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\sum_{w}\frac{(\rho^B, wv)^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\sum_{c,w|h}\frac{1}{h^{t-s-i\epsilon_1 r_1-i\epsilon_2 r_2}}. \tag{5.36}$$

## 5.8 Asymptotics of the off-off-diagonal term

In this section, Theorem 5.0.1 is proved. As a consequence, we obtain an asymptotic formula for $M^{OOD}$ at the critical point.

Let us start with transforming the off-off-diagonal term.

**Proposition 5.8.1.** *One has*

$$M^{OOD} = \sum_{\epsilon_1,\epsilon_2=\pm 1} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)\frac{1}{(2\pi i)^2} \times \int_{\Re t=k/2+0.6}\int_{\Re s=k/2-0.4} E(s,t)\Phi(s,t)2sds2tdt, \quad (5.37)$$

*where*

$$E(s,t) := \hat{q}^{-2t_1-2t_2}\frac{G(s)G(t)}{G(t_1)G(t_2)}\frac{1}{s^2-t_1^2}\frac{1}{t^2-t_2^2} \times \frac{\Gamma(k/2+s+i\epsilon_1 r_1)\Gamma(k/2+t+i\epsilon_2 r_2)\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)\Gamma(k/2+t_2+ir_2)\Gamma(k/2+t_2-ir_2)}, \quad (5.38)$$

$$\Phi(s,t) := 2\zeta_q(1+2t)\frac{q^{s+t}}{(2\pi)^{2t}}\left[\cos(\pi s)+\cos(\pi(t-i\epsilon_1 r_1-i\epsilon_2 r_2))\right] \times \Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)\Gamma(t+s-i\epsilon_1 r_1-i\epsilon_2 r_2)\sum_{A,B=0}^{2} C(A,B)\sum_{q|cp^A} TD(c) \quad (5.39)$$

*and coefficients $C(A,B)$ are given in the table 5.1.*

*Proof.* Consider the term $M^{OOD}(B)$. Möbius function does not vanish only if $(q,g)=1$ or

$(q,g)=\rho$. Then we can write

$$
\begin{aligned}
M^{OOD}(B) = &\frac{2}{(2\pi i)^2}\hat{q}^{-2t_1-2t_2}\sum_{\epsilon_1,\epsilon_2=\pm1}\zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)\\
&\times \int_{\Re t=k/2+0.7}\int_{\Re s=k/2-0.4}\frac{G(s)G(t)}{G(t_1)G(t_2)}\zeta_q(1+2s)\zeta_q(1+2t)\frac{q^{s+t}}{(2\pi)^{2t}}\\
&\times \Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)\Gamma(t+s-i\epsilon_1 r_1-i\epsilon_2 r_2)\\
&\times \frac{\Gamma(k/2+s+i\epsilon_1 r_1)\Gamma(k/2+t+i\epsilon_2 r_2)\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)\Gamma(k/2+t_2+ir_2)\Gamma(k/2+t_2-ir_2)}\\
&\times \sum_{(q,g)=1}\frac{\mu(g)}{g^{1+2s}}\left[\sum_{q|c}TD(c)-(\frac{1}{\rho^{1+2s}}+\frac{1}{\rho})\sum_{q|c\rho}TD(c)+\frac{1}{\rho^{2+2s}}\sum_{q|c\rho^2}TD(c)\right]\\
&\times\left[\cos(\pi s)+\cos(\pi(t-i\epsilon_1 r_1-i\epsilon_2 r_2))\right]\frac{2sds}{s^2-t_1^2}\frac{2tdt}{t^2-t_2^2}.
\end{aligned}
$$

Note that

$$
\zeta_q(1+2s)\sum_{(q,g)=1}\frac{\mu(g)}{g^{1+2s}}=1.
$$

In order to simplify notations, we denote

$$
\begin{aligned}
E(s,t) := &\hat{q}^{-2t_1-2t_2}\frac{G(s)G(t)}{G(t_1)G(t_2)}\frac{1}{s^2-t_1^2}\frac{1}{t^2-t_2^2}\\
&\times\frac{\Gamma(k/2+s+i\epsilon_1 r_1)\Gamma(k/2+t+i\epsilon_2 r_2)\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)\Gamma(k/2+t_2+ir_2)\Gamma(k/2+t_2-ir_2)}.
\end{aligned}
$$

This is an even function since $G$ is even. By equation (4.52)

$$
M^{OOD}=M^{OOD}(0)-\tau_{1/2+ir_2}(\rho)M^{OOD}(1)+M^{OOD}(2).
$$

Next, we introduce parameter $A$ corresponding to the summation condition $q|c\rho^A$. So that

$$
M^{OOD}=\sum_{A,B=0}^{2}C(A,B)M^{OOD}(A,B),
$$

| | $A=0$ | $A=1$ | $A=2$ |
|---|---|---|---|
| $B=0$ | 1 | $-\frac{1+\rho^{2s}}{\rho^{2s+1}}$ | $\frac{1}{\rho^{2+2s}}$ |
| $B=1$ | $-\tau_{1/2+ir_2}(\rho)$ | $\tau_{1/2+ir_2}(\rho)\frac{1+\rho^{2s}}{p^{2s+1}}$ | $-\tau_{1/2+ir_2}(\rho)\frac{1}{\rho^{2+2s}}$ |
| $B=2$ | 1 | $-\frac{1+\rho^{2s}}{\rho^{2s+1}}$ | $\frac{1}{\rho^{2+2s}}$ |

Table 5.1: Values of coefficients $C(A,B)$

$$M^{OOD}(A,B) = \frac{2}{(2\pi i)^2} \sum_{\epsilon_1,\epsilon_2=\pm1} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)$$

$$\times \int_{\Re t=k/2+0.7} \int_{\Re s=k/2-0.4} E(s,t)\zeta_q(1+2t)\frac{q^{s+t}}{(2\pi)^{2t}} \left[\cos(\pi s) + \cos(\pi(t - i\epsilon_1 r_1 - i\epsilon_2 r_2))\right]$$

$$\times \Gamma(t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2)\Gamma(t+s-i\epsilon_1 r_1 - i\epsilon_2 r_2) \sum_{q|c\rho^A} TD(c) 2sds2tdt,$$

where coefficients $C(A,B)$ are given in the table 5.1.

□

The following lemma below will allow us to remove divisibility condition $c, w|h$ in the expression $\sum_{q|c\rho^A} TD(c)$.

**Lemma 5.8.2.**

$$\sum_{\substack{c,w\\ \rho^{\nu-A}|c}} f(c,w) \sum_{c,w|h} g(h) = \sum_{\rho\nmid u} \mu(u) \sum_{\substack{\beta\geqslant 0\\ \delta=\max(\nu-A,\beta)}} \sum_{\substack{c\\ \rho\nmid d\\ \rho\nmid w}} f(\rho^{\nu-A}duc, \rho^{\beta}duw) \sum_h g(\rho^{\delta}u^2 dcwh).$$

**Remark 5.8.3.** *Recall that $q=\rho^{\nu}$, $\nu \geqslant 3$ and so $\nu - A \geqslant 1$.*

*Proof.* Consider

$$S := \sum_{\substack{c,w\\ \rho^{\nu-A}|c}} f(c,w) \sum_{c,w|h} g(h).$$

Let us make the following change of variables

$$c = \rho^{\nu-A} c_1 = \rho^{\nu-A} dc_2,$$

$$w = \rho^{\beta} w_1 = \rho^{\beta} dw_2,$$

$$d = (c_1, w_1) \text{ so that } (c_2, w_2) = 1 \text{ and } \rho \nmid dw_2,$$

$$h = \rho^\delta dc_2w_2h_1 \text{ where } \delta = \max(\nu - A, \beta).$$

Then

$$S = \sum_{\substack{\beta \geqslant 0 \\ \delta = \max(\nu - A, \beta)}} \sum_{\rho \nmid d} \sum_{\substack{c_2 \\ \rho \nmid w_2 \\ (c_2, w_2) = 1}} f(\rho^{\nu - A} dc_2, \rho^\beta dw_2) \sum_{h_1} g(\rho^\delta dc_2 w_2 h_1).$$

Finally, we remove the condition $(c_2, w_2) = 1$ by Möbius inversion

$$S = \sum_{\rho \nmid u} \mu(u) \sum_{\substack{\beta \geqslant 0 \\ \delta = \max(\nu - A, \beta)}} \sum_{\substack{c \\ \rho \nmid d \\ \rho \nmid w}} f(\rho^{\nu - A} duc, \rho^\beta duw) \sum_h g(\rho^\delta u^2 dcwh).$$

□

**Proposition 5.8.4.** *One has*

$$\Phi(s,t) = q^{t-s}(2\pi)^{-2i\epsilon_1 r_1 - 2i\epsilon_2 r_2} \frac{\zeta_q(1+t+s+i\epsilon_1 r_1 + i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1 + i\epsilon_2 r_2)}{\zeta_q(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)}$$

$$\times \zeta(1-t+s+i\epsilon_1 r_1 + i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1 + i\epsilon_2 r_2) \sum_{\alpha \geqslant 0} \frac{\mu(\rho^\alpha)}{\rho^{\alpha(2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)}}$$

$$\times \sum_{A,B=0}^{2} C(A,B)(\rho^A)^{2s} \sum_{\substack{\beta \geqslant 0 \\ \delta = \max(\nu - A, \beta)}} \frac{(\rho^B, \rho^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)} \rho^{\beta(1+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \rho^{\delta(t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2)}}.$$

*Proof.* The expression $TD(c)$ is given by (5.36). Consider

$$\sum_{\rho^{\nu-A}|c} TD(c) = \sum_{\rho^{\nu-A}|c} \frac{1}{c^{2s}} \sum_v \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \sum_w \frac{(\rho^B, wv)^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)} w^{1+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \sum_{c,w|h} \frac{1}{h^{t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2}}.$$

According to Lemma 5.8.2

$$c \to \rho^{\nu - A} duc,$$

$$w \to \rho^\beta duw,$$

$$h \to \rho^\delta u^2 dcwh.$$

Sum over $v$ can be decomposed as

$$\sum_{v} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} = \sum_{\alpha\geqslant 0} \frac{\mu(\rho^\alpha)}{\rho^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \sum_{(v,\rho)=1} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}.$$

Then

$$\sum_{q|c\rho^A} TD(c) = \left(\frac{\rho^A}{q}\right)^{2s} \sum_{(u,\rho)=1} \frac{\mu(u)}{u^{2t+1}} \sum_{(v,\rho)=1} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}$$
$$\times \sum_{c} \frac{1}{c^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2}} \sum_{h} \frac{1}{h^{t-s-i\epsilon_1 r_1-i\epsilon_2 r_2}} \sum_{(d,\rho)=1} \frac{1}{d^{1+t+s+i\epsilon_1 r_+ i\epsilon_2 r_2}} \sum_{(w,\rho)=1} \frac{1}{w^{1+t-s+i\epsilon_1 r_+ i\epsilon_2 r_2}}$$
$$\times \sum_{\alpha\geqslant 0} \frac{\mu(\rho^\alpha)}{\rho^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \sum_{\substack{\beta\geqslant 0\\ \delta=\max(\nu-A,\beta)}} \frac{(\rho^B,\rho^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}\rho^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}\rho^{\delta(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)}}.$$

The asymmetric functional equation (B.3) implies

$$\frac{\Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)\Gamma(t+s-i\epsilon_1 r_1-i\epsilon_2 r_2)\prod\zeta(t\pm s-i\epsilon_1 r_1-i\epsilon_2 r_2)}{(2\pi)^{2t-2i\epsilon_1 r_1-2i\epsilon_2 r_2}}$$
$$= \frac{\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{2\left[\cos(\pi s)+\cos(\pi(t-i\epsilon_1 r_1-i\epsilon_2 r_2))\right]}.$$

Thus,

$$\Phi(s,t) = q^{t-s}(2\pi)^{-2i\epsilon_1 r_1-2i\epsilon_2 r_2} \frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}$$
$$\times \zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \sum_{\alpha\geqslant 0} \frac{\mu(\rho^\alpha)}{\rho^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}$$
$$\times \sum_{A,B=0}^{2} C(A,B)(\rho^A)^{2s} \sum_{\substack{\beta\geqslant 0\\ \delta=\max(\nu-A,\beta)}} \frac{(\rho^B,\rho^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}\rho^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}\rho^{\delta(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)}}.$$

$\square$

Sums over $\alpha$ and $\beta$ in $\Phi(s,t)$ can be evaluated by considering different cases.

### 5.8.1 Case 1: $\beta > \nu - A$

**Proposition 5.8.5.** *The given case contributes as an error term to $M^{OOD}$.*

*Proof.* We have $\delta = \beta$ and

$$\Phi(s,t) = q^{t-s}(2\pi)^{-2i\epsilon_1 r_1 - 2i\epsilon_2 r_2}\frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}$$
$$\times \zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\sum_{\alpha\geqslant 0}\frac{\mu(\rho^\alpha)}{\rho^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}$$
$$\times \sum_{A,B=0}^{2} C(A,B)(\rho^A)^{2s}\sum_{\beta\geqslant \nu-A+1}\frac{(\rho^B,\rho^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}\rho^{\beta(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}}.$$

Consider the sum over $\beta$.

$$q^{t-s}\sum_{\beta\geqslant \nu-A+1}\frac{1}{(\rho^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2})^\beta} = \frac{1}{q}(\rho^{A-1})^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2}\sum_{\beta\geqslant 0}\frac{1}{(\rho^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2})^\beta}.$$

This implies that the contribution of this case to $M^{OOD}$ is bounded by

$$P(r_1)P(r_2)q^{-1-t_1-t_2+\epsilon}.$$

□

### 5.8.2 Case 2: $\beta \leqslant \nu - A$

Condition $\beta \leqslant \nu - A$ means that $\delta = \nu - A$ and

$$\Phi(s,t) = \hat{q}^{2i\epsilon_1 r_1+2i\epsilon_2 r_2}\frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}$$
$$\times \zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\sum_{\alpha\geqslant 0}\frac{\mu(\rho^\alpha)}{\rho^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}$$
$$\times \sum_{A,B=0}^{2} C(A,B)(\rho^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_r}\sum_{0\leqslant\beta\leqslant \nu-A}\frac{(\rho^B,\rho^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}\rho^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}.$$

The sum over $\beta$ can be decomposed in the following way:

$$\sum_{0\leqslant\beta\leqslant\nu-A} \frac{(\rho^B,\rho^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}\rho^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}$$

$$= \sum_{\substack{0\leqslant\beta\leqslant\nu-A\\ B\leqslant\alpha+\beta}} \frac{(\rho^B)^{i\epsilon_2 r_2}}{\rho^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} + \sum_{\substack{0\leqslant\beta\leqslant\nu-A\\ B>\alpha+\beta}} \frac{(\rho^\alpha)^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}\rho^{\beta(2i\epsilon_1 r_1)}}$$

$$= \sum_{0\leqslant\beta\leqslant\nu-A} \frac{(\rho^B)^{i\epsilon_2 r_2}}{\rho^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} - \sum_{\substack{0\leqslant\beta\leqslant\nu-A\\ B>\alpha+\beta}} \frac{(\rho^B)^{i\epsilon_2 r_2}}{\rho^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} + \sum_{\substack{0\leqslant\beta\leqslant\nu-A\\ B>\alpha+\beta}} \frac{(\rho^\alpha)^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}\rho^{\beta(2i\epsilon_1 r_1)}}.$$

The first sum does not contribute to $\Phi(s,t)$ since

$$\sum_{0\leqslant\beta\leqslant\nu-A} \frac{1}{\rho^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} = \left(1-\frac{1}{\rho^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\right)^{-1}\left(1+O\left(\frac{1}{q}\right)\right)$$

and

$$\sum_{A,B=0}^{2} C(A,B)(\rho^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2}(\rho^B)^{i\epsilon_2 r_2} = 0.$$

Therefore,

$$\Phi(s,t) = \hat{q}^{2i\epsilon_1 r_1+2i\epsilon_2 r_2}\frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}$$

$$\times\,\zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\sum_{A,B=0}^{2} C(A,B)(\rho^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_r}$$

$$\times\sum_{\alpha\geqslant 0}\frac{\mu(\rho^\alpha)}{\rho^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}\sum_{\substack{0\leqslant\beta\leqslant\nu-A\\ B>\alpha+\beta}}\left(\frac{-(\rho^B)^{i\epsilon_2 r_2}}{\rho^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}+\frac{(\rho^\alpha)^{1+2i\epsilon_2 r_2}}{\rho^{B(1+i\epsilon_2 r_2)}\rho^{\beta(2i\epsilon_1 r_1)}}\right).$$

For each fixed $B$ the sum over $A$ can be evaluated using the table 5.1:

$$\sum_{A=0}^{2} C(A,0)(\rho^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} = (1-\rho^{t+s+1-i\epsilon_1 r_1-i\epsilon_2 r_2})(1-\rho^{t-s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}),$$

$$\sum_{A=0}^{2} C(A,1)(\rho^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} = -(\rho^{ir_2}+\rho^{-ir_2})(1-\rho^{t+s+1-i\epsilon_1 r_1-i\epsilon_2 r_2})(1-\rho^{t-s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}),$$

$$\sum_{A=0}^{2} C(A,2)(\rho^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} = (1-\rho^{t+s+1-i\epsilon_1 r_1-i\epsilon_2 r_2})(1-\rho^{t-s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}).$$

Since $B=0,1,2$, the condition $B>\alpha+\beta$ is satisfied in four cases

$$(B,\alpha,\beta)=\{(1,0,0),(2,0,0),(2,1,0),(2,0,1)\}.$$

Thus,

$$\Phi(s,t) = \hat{q}^{2i\epsilon_1 r_1+2i\epsilon_2 r_2}\frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}$$
$$\times\, \zeta_q(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)$$
$$\times \Bigg[(\rho^{ir_2}+\frac{1}{\rho^{ir_2}})(\rho^{i\epsilon_2 r_2}-\frac{1}{\rho^{1+i\epsilon_2 r_2}})-\rho^{2i\epsilon_2 r_2}+\frac{1}{\rho^{2+2i\epsilon_2 r_2}}$$
$$+\frac{1}{\rho^{2+2i\epsilon_1 r_1}}-\frac{1}{\rho^{3+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}-\frac{1}{\rho^{1+2i\epsilon_1 r_1}}+\frac{1}{\rho^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\Bigg].$$

Simplifying, we have

$$\Phi(s,t) = \frac{\phi(q)}{q}\hat{q}^{2i\epsilon_1 r_1+2i\epsilon_2 r_2}\frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}$$
$$\times\, \zeta_q(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\left(1-\frac{1}{\rho^{1+2i\epsilon_1 r_1}}\right)\left(1-\frac{1}{\rho^{1+2i\epsilon_2 r_2}}\right).$$

### *5.8.3 Proof of Theorem 5.0.1*

By (5.37) and (5.38), we have

$$M^{OOD} = \frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm1}\frac{\zeta_q(1+2i\epsilon_1 r_1)\zeta_q(1+2i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}$$
$$\times\, \hat{q}^{-2t_1-2t_2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}\frac{1}{(2\pi i)^2}\int_{\Re t=k/2+0.7}\int_{\Re s=k/2-0.4} I_{\epsilon_1,\epsilon_2}(s,t)\frac{2sds}{s^2-t_1^2}\frac{2tdt}{t^2-t_2^2},$$

where

$$I_{\epsilon_1,\epsilon_2}(s,t) = \frac{G(s)G(t)}{G(t_1)G(t_2)}\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)$$
$$\times \zeta_q(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)$$
$$\times \frac{\Gamma(k/2+s+i\epsilon_1 r_1)\Gamma(k/2+t+i\epsilon_2 r_2)\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)\Gamma(k/2+t_2+ir_2)\Gamma(k/2+t_2-ir_2)}.$$

The function $I_{\epsilon_1,\epsilon_2}(s,t)$ is even in both $s$ and $t$. Therefore,

$$4\frac{1}{(2\pi i)^2}\int_{\Re t=k/2+0.7}\int_{\Re s=k/2-0.4} I_{\epsilon_1,\epsilon_2}(s,t)\frac{2sds}{s^2-t_1^2}\frac{2tdt}{t^2-t_2^2}$$
$$= \mathrm{Res}_{s=t_1,t=t_2}\, I(s,t)\frac{2s}{s^2-t_1^2}\frac{2t}{t^2-t_2^2} + \mathrm{Res}_{s=t_1,t=-t_2}\, I(s,t)\frac{2s}{s^2-t_1^2}\frac{2t}{t^2-t_2^2}$$
$$+ \mathrm{Res}_{s=-t_1,t=t_2}\, I(s,t)\frac{2s}{s^2-t_1^2}\frac{2t}{t^2-t_2^2} + \mathrm{Res}_{s=-t_1,t=-t_2}\, I(s,t)\frac{2s}{s^2-t_1^2}\frac{2t}{t^2-t_2^2}.$$

Each of the four given residues has the same value. Consequently,

$$M^{OOD} = \frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm 1}\frac{\zeta_q(1+2i\epsilon_1 r_1)\zeta_q(1+2i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}\hat{q}^{-2t_1-2t_2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}$$
$$\times \zeta_q(1+t_1+t_2+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1+t_2-t_1+i\epsilon_1 r_1+i\epsilon_2 r_2)$$
$$\times\zeta_q(1+t_1-t_2+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1-t_1-t_2+i\epsilon_1 r_1+i\epsilon_2 r_2)\frac{\Gamma(k/2-t_1+i\epsilon_1 r_1)\Gamma(k/2-t_2+i\epsilon_2 r_2)}{\Gamma(k/2+t_1-i\epsilon_1 r_1)\Gamma(k/2+t_2-i\epsilon_2 r_2)}.$$

### *5.8.4 The off-off-diagonal term at the critical point*

**Theorem 5.8.6.** *For any $\epsilon > 0$, up to an error $O_{\epsilon,\rho}(q^{\epsilon}(q^{-\frac{2k-3}{12}} + q^{-1/4}))$, we have*

$$M^{OOD}(0,0,0,0) = \lim_{r_1\to 0}\lim_{r_1\to 0}\lim_{t_1\to 0}\lim_{t_2\to 0} M^{OOD}$$
$$= \frac{1}{(2\pi i)^2}\int_{\Re t=k/2+0.7}\int_{\Re s=k/2-0.4} g(s,t)\frac{2ds}{s}\frac{2dt}{t}, \quad (5.40)$$

*where*

$$g(s,t) = \left(\frac{\phi(q)}{q}\right)^3 \frac{1}{\zeta_q(2)} \frac{G(s)G(t)}{G(0)^2} \prod \zeta_q(1 \pm t \pm s) \frac{\Gamma(k/2+s)\Gamma(k/2+t)\Gamma(k/2-s)\Gamma(k/2-t)}{\Gamma(k/2)^4}$$

$$\times \Bigg[ (2\log \hat{q} + \gamma)^2 + \sum \frac{\zeta_q''}{\zeta_q}(1 \pm t \pm s) + 2 \sum^{*} \frac{\zeta_q'}{\zeta_q}(1 \pm t \pm s) \frac{\zeta_q'}{\zeta_q}(1 \pm t \pm s)$$

$$+ (2\log \hat{q} + \gamma) \left( 4\frac{\zeta_q'}{\zeta_q}(2) - 2\sum \frac{\zeta_q'}{\zeta_q}(1 \pm t \pm s) - \sum \frac{\Gamma'}{\Gamma}(k/2 \pm s) - \sum \frac{\Gamma'}{\Gamma}(k/2 \pm t) \right)$$

$$+ \sum \frac{\zeta_q'}{\zeta_q}(1 \pm t \pm s) \left( -4\frac{\zeta_q'}{\zeta_q}(2) + \sum \frac{\Gamma'}{\Gamma}(k/2 \pm t) + \sum \frac{\Gamma'}{\Gamma}(k/2 \pm s) \right) + \sum \frac{\Gamma'}{\Gamma}(k/2 \pm t) \sum \frac{\Gamma'}{\Gamma}(k/2 \pm s)$$

$$-4\frac{\zeta_q''}{\zeta_q}(2) + 8\left(\frac{\zeta_q'}{\zeta_q}(2)\right)^2 - 2\frac{\zeta_q'}{\zeta_q}(2)\left(\sum \frac{\Gamma'}{\Gamma}(k/2 \pm t) \sum \frac{\Gamma'}{\Gamma}(k/2 \pm s)\right) \Bigg].$$

*Here*

$$\sum^{*} \frac{\zeta_q'}{\zeta_q}(1 \pm t \pm s) \frac{\zeta_q'}{\zeta_q}(1 \pm t \pm s) = \sum_{\substack{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 = \pm 1 \\ (\epsilon_1,\epsilon_2) \neq (\epsilon_3,\epsilon_4)}} \frac{\zeta_q'}{\zeta_q}(1 + \epsilon_1 t + \epsilon_2 s) \frac{\zeta_q'}{\zeta_q}(1 + \epsilon_3 t + \epsilon_4 s).$$

**Corollary 5.8.7.** *$M^{OOD}(0,0,0,0)$ is a polynomial in $\log q$ of order 2.*

*Proof.* First, we let $t_1, t_2 \to 0$. Then

$$M^{OOD}(0,0,r_1,r_2) := \lim_{t_1 \to 0} \lim_{t_2 \to 0} M^{OOD}(t_1,t_2,r_1,r_2) =$$

$$\frac{\phi(q)}{q} \sum_{\epsilon_1,\epsilon_2 = \pm 1} \frac{\zeta_q(1+2i\epsilon_1 r_1)\zeta_q(1+2i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \hat{q}^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}$$

$$\times \frac{1}{(2\pi i)^2} \int_{\Re t = k/2+0.7} \int_{\Re s = k/2-0.4} I_{\epsilon_1,\epsilon_2}(s,t) \frac{2ds}{s} \frac{2dt}{t},$$

where

$$I_{\epsilon_1,\epsilon_2}(s,t) = \frac{G(s)G(t)}{G(0)^2}\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)$$

$$\times\,\zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)$$

$$\times\,\frac{\Gamma(k/2+s+i\epsilon_1 r_1)\Gamma(k/2+t+i\epsilon_2 r_2)\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+ir_1)\Gamma(k/2-ir_1)\Gamma(k/2+ir_2)\Gamma(k/2-ir_2)}.$$

Let

$$f(r_1,r_2) := \frac{\phi(q)}{q}\frac{1}{\zeta_q(2+2ir_1+2ir_2)}\frac{G(s)G(t)}{G(0)^2}$$

$$\times\,\zeta_q(1+t+s+ir_1+ir_2)\zeta_q(1+t-s+ir_1+ir_2)\zeta_q(1-t+s+ir_1+ir_2)\zeta_q(1-t-s+ir_1+ir_2)$$

$$\times\,\frac{\Gamma(k/2+s+ir_1)\Gamma(k/2+t+ir_2)\Gamma(k/2-s+ir_1)\Gamma(k/2-t+ir_2)}{\Gamma(k/2+ir_1)\Gamma(k/2-ir_1)\Gamma(k/2+ir_2)\Gamma(k/2-ir_2)}.$$

Consider

$$g(s,t) := \lim_{r_1\to 0}\lim_{r_2\to 0}\sum_{\epsilon_1,\epsilon_2=\pm 1}\zeta_q(1+2i\epsilon_1 r_1)\zeta_q(1+2i\epsilon_2 r_2)\hat{q}^{2i\epsilon_1 r_1+2i\epsilon_2 r_2}f(\epsilon_1 r_1,\epsilon_2 r_2) =$$

$$\left(\frac{\phi(q)}{q}\right)^2\left[(2\log\hat{q}+\gamma)^2 f(0,0)+i(2\log\hat{q}+\gamma)\left(\frac{\partial f}{\partial r_1}(0,0)+\frac{\partial f}{\partial r_2}(0,0)\right)-\frac{\partial^2 f}{\partial r_1\partial r_2}(0,0)\right].$$

Here

$$\frac{\partial f}{\partial r_1}(0,0) = -if(0,0)\left(2\frac{\zeta_q'}{\zeta_q}(2)-\sum\frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)-\sum\frac{\Gamma'}{\Gamma}(k/2\pm s)\right),$$

$$\frac{\partial f}{\partial r_2}(0,0) = -if(0,0)\left(2\frac{\zeta_q'}{\zeta_q}(2)-\sum\frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)-\sum\frac{\Gamma'}{\Gamma}(k/2\pm t)\right),$$

$$\frac{\partial^2 f}{\partial r_1 \partial r_2}(0,0) = -f(0,0)\left[\sum \frac{\zeta_q''}{\zeta_q}(1\pm t\pm s) + 2\sum^{*} \frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)\frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)\right.$$

$$+\sum \frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)\left(-4\frac{\zeta_q'}{\zeta_q}(2) + \sum \frac{\Gamma'}{\Gamma}(k/2\pm t) + \sum \frac{\Gamma'}{\Gamma}(k/2\pm s)\right) + \sum \frac{\Gamma'}{\Gamma}(k/2\pm t)\sum \frac{\Gamma'}{\Gamma}(k/2\pm s)$$

$$\left. -4\frac{\zeta_q''}{\zeta_q}(2) + 8\left(\frac{\zeta_q'}{\zeta_q}(2)\right)^2 - 2\frac{\zeta_q'}{\zeta_q}(2)\left(\sum \frac{\Gamma'}{\Gamma}(k/2\pm t)\sum \frac{\Gamma'}{\Gamma}(k/2\pm s)\right)\right].$$

Then

$$M^{OOD}(0,0,0,0) = \frac{1}{(2\pi i)^2}\int_{\Re t = k/2+0.7}\int_{\Re s = k/2-0.4} g(s,t)\frac{2ds}{s}\frac{2dt}{t}$$

with

$$g(s,t) = \left(\frac{\phi(q)}{q}\right)^3 \frac{1}{\zeta_q(2)}\frac{G(s)G(t)}{G(0)^2}\prod \zeta_q(1\pm t\pm s)\frac{\Gamma(k/2+s)\Gamma(k/2+t)\Gamma(k/2-s)\Gamma(k/2-t)}{\Gamma(k/2)^4}$$

$$\times\left[(2\log\hat{q}+\gamma)^2 + \sum \frac{\zeta_q''}{\zeta_q}(1\pm t\pm s) + 2\sum^{*} \frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)\frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)\right.$$

$$+ (2\log\hat{q}+\gamma)\left(4\frac{\zeta_q'}{\zeta_q}(2) - 2\sum \frac{\zeta_q'}{\zeta_q}(1\pm t\pm s) - \sum \frac{\Gamma'}{\Gamma}(k/2\pm s) - \sum \frac{\Gamma'}{\Gamma}(k/2\pm t)\right)$$

$$+\sum \frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)\left(-4\frac{\zeta_q'}{\zeta_q}(2) + \sum \frac{\Gamma'}{\Gamma}(k/2\pm t) + \sum \frac{\Gamma'}{\Gamma}(k/2\pm s)\right) + \sum \frac{\Gamma'}{\Gamma}(k/2\pm t)\sum \frac{\Gamma'}{\Gamma}(k/2\pm s)$$

$$\left. -4\frac{\zeta_q''}{\zeta_q}(2) + 8\left(\frac{\zeta_q'}{\zeta_q}(2)\right)^2 - 2\frac{\zeta_q'}{\zeta_q}(2)\left(\sum \frac{\Gamma'}{\Gamma}(k/2\pm t)\sum \frac{\Gamma'}{\Gamma}(k/2\pm s)\right)\right].$$

The function $g(s,t)$ is even in both variables $s$ and $t$. Therefore,

$$M^{OOD}(0,0,0,0) = \frac{1}{4}\operatorname{Res}_{s=t=0}\frac{4g(s,t)}{st} = \operatorname{Res}_{t=0}\frac{g(0,t)}{t}.$$

To find the order of leading term, we replace all $\zeta(1\pm t)$ by $\frac{1}{\pm t}$. Let

$$r(t) := \frac{G(t)}{G(0)}\frac{\Gamma(k/2+t)\Gamma(k/2-t)}{\Gamma(k/2)^2},$$

then

$$\left(\frac{\phi(q)}{q}\right)^7 \frac{1}{\zeta_q(2)} \operatorname{Res}_{t=0} \frac{r(t)}{t^5}\left((\log q)^2+\frac{4}{t^2}\right)=\left(\frac{\phi(q)}{q}\right)^7 \frac{1}{\zeta_q(2)} \frac{1}{6!}(4r^{(6)}(0)+30r^{(4)}(0)(\log q)^2).$$

Therefore, the $M^{OOD}(0,0,0,0)$ is a polynomial in $\log q$ of order 2.

$\square$

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# Appendix A

## Gamma function

Let $n$ be a positive integer, then

$$\Gamma(n) = (n-1)! \tag{A.1}$$

For complex numbers with a positive real part we define $\Gamma$ via integral

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx. \tag{A.2}$$

By analytic continuation it can be extended to all complex numbers except the non-positive integers, where the function has simple poles with residue

$$\operatorname{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}. \tag{A.3}$$

There are no points $z \in \mathbb{C}$ at which $\Gamma(z) = 0$.

Gamma function satisfies the functional equation

$$\Gamma(z+1) = z\Gamma(z). \tag{A.4}$$

**Lemma A.0.8.** *(Stirling's formula, [4] p. 52)*

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z (1 + O(1/z)). \tag{A.5}$$

**Lemma A.0.9.** *([31], 5.11.9) As $y \to \pm\infty$, we have*

$$|\Gamma(x+iy)| \sim \sqrt{2\pi}|y|^{x-1/2}e^{-\pi|y|/2}. \tag{A.6}$$

**Lemma A.0.10.** *([31], 5.11.12) If $z \to \infty$ in the sector $|\arg z| < \pi$, then*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}. \tag{A.7}$$

**Lemma A.0.11.** *For $|z_j - 1| < 1$, we have*

$$\Gamma(z) = \frac{1}{z} - \gamma + O(z). \tag{A.8}$$

*Proof.* This follows from Laurent expansion of $\Gamma(z)$ at 1. □

**Lemma A.0.12.** *(Euler's reflection formula, [4] p. 51)*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{A.9}$$

**Lemma A.0.13.** *(Duplication formula, [4] p. 52)*

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z). \tag{A.10}$$

# Appendix B

## Riemann zeta function

We define the Riemann zeta function as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},\ \Re s > 1. \tag{B.1}$$

It has an Euler product

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}. \tag{B.2}$$

**Lemma B.0.14.** *(Asymmetric functional equation, [29], p. 121) Zeta is an analytic function on $\mathbb{C} \setminus \{1\}$ and satisfies there the functional equation*

$$\frac{\Gamma(t)\zeta(t)}{(2\pi)^t} = \frac{\zeta(1-t)}{2\cos(\pi t/2)}. \tag{B.3}$$

**Lemma B.0.15.** *([33], p. 90) As $s \to 1$,*

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n. \tag{B.4}$$

**Theorem B.0.16.** *([21], p. 116-117)*

*Let $s = \sigma + it$.*

*For $|t| \geqslant 2$ one has*

$$\zeta(1+it) = O(\log^{2/3}|t|). \tag{B.5}$$

*There exists an absolute constant $\gamma_1 > 0$ such that for*

$$\sigma \geqslant 1 - \frac{\gamma_1}{\log^{2/3}|t|},\ |t| \geqslant 2 \tag{B.6}$$

*we have*

$$\zeta(\sigma+it) = O(\log^{2/3}|t|). \tag{B.7}$$

*For $1/2 \leqslant \sigma \leqslant 1$, $|t| \geqslant 2$ there exists an absolute constant $a > 0$ such that*

$$\zeta(s) = O(|t|^{a(1-\sigma)^{3/2}}\log|t|). \tag{B.8}$$

# Appendix C

## Bessel functions

We define the *Bessel function of the first kind* by

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu}. \tag{C.1}$$

The *Bessel function of the second kind* can be expressed in terms of $J_\nu(z)$ as follows

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin(\pi\nu)}. \tag{C.2}$$

And the *modified Bessel function of the second kind* is given by the integral formula

$$K_\nu(z) = \frac{\Gamma(\nu+1/2)(2z)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos t dt}{(t^2+z^2)^{\nu+1/2}}. \tag{C.3}$$

**Lemma C.0.17.** *([26], Lemma C.1) Let $z > 0$ and $v \in \mathbb{C}$, then*

$$\left(z^v J_v(z)\right)' = z^v J_{v-1}(z), \tag{C.4}$$

$$\left(z^v Y_v(z)\right)' = z^v Y_{v-1}(z), \tag{C.5}$$

$$\left(z^v K_v(z)\right)' = -z^v K_{v-1}(z). \tag{C.6}$$

**Lemma C.0.18.** *([26], Lemma C.2) For $z > 0$ and $j \geqslant 0$ we have*

$$\frac{z^j}{(1+z)^j} J_v^{(j)}(z) \ll_{j,v} \frac{z^{\Re v}}{(1+z)^{\Re v + 1/2}}, \tag{C.7}$$

$$\frac{z^j}{(1+z)^j} Y_0^{(j)}(z) \ll_j \frac{(1+|\log z|)}{(1+z)^{1/2}}, \tag{C.8}$$

$$\frac{z^j}{(1+z)^j} K_v^{(j)}(z) \ll_{j,v} \frac{e^{-z}(1+|\log z|)}{(1+z)^{1/2}} \text{ if } \Re v = 0. \tag{C.9}$$

**Lemma C.0.19.** *([41], page 149) Assume that $\Re(\mu_1 + \mu_2 + 1) > \Re(2s) > 0$, then*

$$\begin{aligned}\int_0^\infty \frac{J_{\mu_1}(z) J_{\mu_2}(z)}{z^{2s}} dz &= \frac{1}{2^{2s}} \\ &\times \frac{\Gamma(2s)\Gamma(\mu_1/2 + \mu_2/2 - s + 1/2)}{\Gamma(-\mu_1/2 + \mu_2/2 + s + 1/2)\Gamma(\mu_1/2 + \mu_2/2 + s + 1/2)\Gamma(\mu_1/2 - \mu_2/2 + s + 1/2)}.\end{aligned} \tag{C.10}$$

**Lemma C.0.20.** *Assume that $k$ is an even integer, then*

$$\int_0^\infty Y_0(z) J_{k-1}(z) z^{-2s} dz = (-1)^{1+k/2} \frac{1}{2\sqrt{\pi}} \frac{\Gamma(s)\Gamma^2(k/2 - s)}{\Gamma^2(k/2 + s)\Gamma(1/2 - s)}. \tag{C.11}$$

*Proof.* According to Watson's book [41] (page 149)

$$\pi J_{\mu_1}(z) Y_{\mu_2}(z) = \frac{\partial}{\partial v}\{J_{\mu_1}(z) J_v(z)\} - (-1)^{\mu_2} \frac{\partial}{\partial v}\{J_{\mu_1}(z) J_{-v}(z)\},$$

where we let $v := \mu_2$ after the differention is done. Therefore, by lemma C.0.19

$$\int_0^\infty J_{k-1}(z) Y_0(z) z^{-2s} dz = 1/\pi \frac{\partial}{\partial v}\left(\int_0^\infty (J_{k-1}(z) J_v(z) - J_{k-1}(z) J_{-v}(z)) \frac{dz}{z^{2s}}\right)$$

$$= \frac{\Gamma(2s)}{2^{2s}\pi} \frac{\partial}{\partial v}\left(f_1(v) - f_2(v)\right),$$

where

$$f_1(v) = \frac{\Gamma(k/2 + v/2 - s)}{\Gamma(-k/2 + v/2 + s + 1)\Gamma(k/2 + v/2 + s)\Gamma(k/2 - v/2 + s)}$$

$$f_2(v) = f_1(-v) = \frac{\Gamma(k/2 - v/2 - s)}{\Gamma(-k/2 - v/2 + s + 1)\Gamma(k/2 - v/2 + s)\Gamma(k/2 + v/2 + s)}.$$

Differentiating and letting $v := \mu_2 = 0$, we have

$$\int_0^\infty J_{k-1}(z)Y_0(z)z^{-2s}dz =$$

$$\frac{\Gamma(2s)}{2^{2s}\pi} \frac{\Gamma(k/2 - s)}{\Gamma^2(k/2 + s)\Gamma(-k/2 + s + 1)} \left(\psi(k/2 - s) - \psi(-k/2 + s + 1)\right).$$

By the duplication formula

$$2^{1-2s}\sqrt{\pi}\Gamma(2s) = \Gamma(s)\Gamma(s + 1/2)$$

and the reflection formula for polygamma function

$$\psi(1 - z) - \psi(z) = \pi \cot(\pi z)$$

we find

$$\int_0^\infty J_{k-1}(z)Y_0(z)z^{-2s}dz = -\frac{1}{2\sqrt{\pi}} \frac{\Gamma(s)\Gamma(s + 1/2)\Gamma(k/2 - s)}{\Gamma^2(k/2 + s)\Gamma(-k/2 + s + 1)} \cot(\pi(k/2 - s)).$$

The reflection formula for the gamma function gives

$$\frac{\Gamma(s + 1/2)}{\Gamma(s - k/2 + 1)} = \frac{\Gamma(k/2 - s)}{\Gamma(1/2 - s)} \frac{\sin(\pi(k/2 - s))}{\sin(\pi/2 + \pi s)}.$$

By our assumption $k$ is an even number. Thus

$$-\cot(\pi(k/2 - s)) \frac{\sin(\pi(k/2 - s))}{\sin(\pi/2 + \pi s)} = -\frac{\cos(\pi(k/2 - s))}{\cos(\pi s)} = (-1)^{1+k/2}.$$

Finally,

$$\int_0^\infty Y_0(z)J_{k-1}(z)z^{-2s}dz = (-1)^{1+k/2} \frac{1}{2\sqrt{\pi}} \frac{\Gamma(s)\Gamma^2(k/2 - s)}{\Gamma^2(k/2 + s)\Gamma(1/2 - s)}.$$

□

**Lemma C.0.21.** *([5], lemma 3) Let $F : (0, \infty) \to \mathbb{C}$ be a smooth function of compact support. For $s \in \mathbb{C}$ let $B_s$ denote one of $J_s$, $Y_s$ or $K_s$. Then for $\alpha > 0$ and $j \in \mathbb{N}$ we have*

$$\int_0^\infty F(x) B_s(\alpha\sqrt{x}) dx = \pm \left(\frac{2}{\alpha}\right)^j \int_0^\infty \frac{\partial^j}{\partial x^j}(F(x)x^{-s/2}) x^{\frac{s+j}{2}} B_{s+j}(\alpha\sqrt{x}) dx. \tag{C.12}$$

**Lemma C.0.22.** *([31], equations 10.6.7 and 10.29.5) For $k = 0, 1, 2, \ldots,$*

$$J_s^{(k)}(z) = \frac{1}{2^k} \sum_{n=0}^{k} (-1)^n \binom{k}{n} J_{s-k+2n}(z), \tag{C.13}$$

$$e^{s\pi i} K_s^{(k)}(z) = \frac{1}{2^k} \Bigg( e^{(s-k)\pi i} K_{s-k}(z) + \binom{k}{1} e^{(s-k+2)\pi i} K_{s-k+2}(z)$$
$$+ \binom{k}{2} e^{(s-k+4)\pi i} K_{s-k+4}(z) + \ldots + e^{(s+k)\pi i} K_{s+k}(z) \Bigg). \tag{C.14}$$

# Appendix D

## Combinatorial identities

**Lemma D.0.23.** *(Leibniz rule, [32], p. 318) Fot the nth derivative of an arbitrary number of factors*

$$\left(\prod_{i=1}^{k} f_i\right)^{(n)}(x) = \sum_{j_1+j_2+\ldots+j_k=n} \binom{n}{j_1, j_2, \ldots, j_k} \prod_{i=1}^{k} f_i^{j_i}(x). \tag{D.1}$$

**Lemma D.0.24.** *(Faá di Bruno's formula, [13], [14], [19]) Suppose $f(x)$ and $g(x)$ are $n$ times differentiable functions. Then*

$$\frac{d^n}{dx^n} f(g(x)) = \sum \binom{n}{m_1, m_2, \ldots, m_n} f^{(m_1+m_2+\ldots+m_n)}(g(x)) \cdot \prod_{j=1}^{n} \left(\frac{g^{(j)}(x)}{j!}\right)^{m_j}, \tag{D.2}$$

*where the sum is over all $n$-tuples $(m_1, m_2, \ldots, m_n)$ such that $1 \cdot m_1 + 2 \cdot m_2 + \ldots n \cdot m_n = n$.*

**Theorem D.0.25.** *(Gauss hypergeometric identity, [2]) Let*

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

*with*

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}.$$

*Then*

$$_2F_1(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

*for* $\Re(c-a-b) > 0$.

# Appendix E

## Mellin transforms

**Lemma E.0.26.** *([30], 2.19, page 15) Let* $\phi(x) = (b + ax)^{-v}$*. Then for* $0 < \Re z < v$

$$\int_0^\infty \phi(x) x^{z-1} dx = (b/a)^z b^{-v} \frac{\Gamma(z)\Gamma(v-z)}{\Gamma(v)}. \tag{E.1}$$

**Lemma E.0.27.** *([30], 2.20, page 16) Let* $\Re v > -1$ *and*

$$\phi(x) = \begin{cases} (a-x)^v & \textit{if } x < a \\ 0 & \textit{if } x > a \end{cases}.$$

*Then for* $\Re z > 0$

$$\int_0^\infty \phi(x) x^{z-1} dx = a^{v+z} \frac{\Gamma(v+1)\Gamma(z)}{\Gamma(v+z+1)}. \tag{E.2}$$

**Lemma E.0.28.** *([30], 2.21, page 16) Let* $\Re v > -1$ *and*

$$\phi(x) = \begin{cases} (x-a)^v & \textit{if } x > a \\ 0 & \textit{if } x < a \end{cases}.$$

*Then for* $\Re z < -\Re v$

$$\int_0^\infty \phi(x)x^{z-1}dx = a^{v+z}\frac{\Gamma(-v-z)\Gamma(v+1)}{\Gamma(1-z)}. \tag{E.3}$$

**Lemma E.0.29.** *([3], p.21) For* $x > 0$

$$J_{k-1}(x) = \frac{1}{2}\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{x}{2}\right)^{-s}\frac{\Gamma(s/2+k/2-1/2)}{\Gamma(-s/2+k/2+1/2)}ds, \tag{E.4}$$

*where* $-k+1 < \sigma < 1$.

Changing variable $-s := k-1+2z$, we obtain

**Lemma E.0.30.** *For* $-k/2 < \sigma < 0$

$$J_{k-1}(x) = -\frac{1}{2\pi i}\int_{(\sigma)}\frac{\pi}{\Gamma(1+z)\Gamma(k+z)\sin(\pi z)}\left(\frac{x}{2}\right)^{k-1+2z}dz. \tag{E.5}$$

Let $\Re v = 1/2$ and $x > 0$. We set

$$k_0(x,v) := \frac{1}{2\cos\pi v}(J_{2v-1}(x) - J_{1-2v}(x)), \tag{E.6}$$

$$k_1(x,v) := \frac{2}{\pi}\sin\pi v K_{2v-1}(x) \tag{E.7}$$

and

$$\gamma(u,v) := \frac{2^{2u-1}}{\pi}\Gamma(u+v-1/2)\Gamma(u-v+1/2). \tag{E.8}$$

**Lemma E.0.31.** *([27], p. 89) One has*

$$\int_0^\infty k_0(x,v)x^{w-1}dx = \gamma(w/2,v)\cos(\pi w/2), \tag{E.9}$$

$$\int_0^\infty k_1(x,v)x^{w-1}dx = \gamma(w/2,v)\sin(\pi v). \tag{E.10}$$

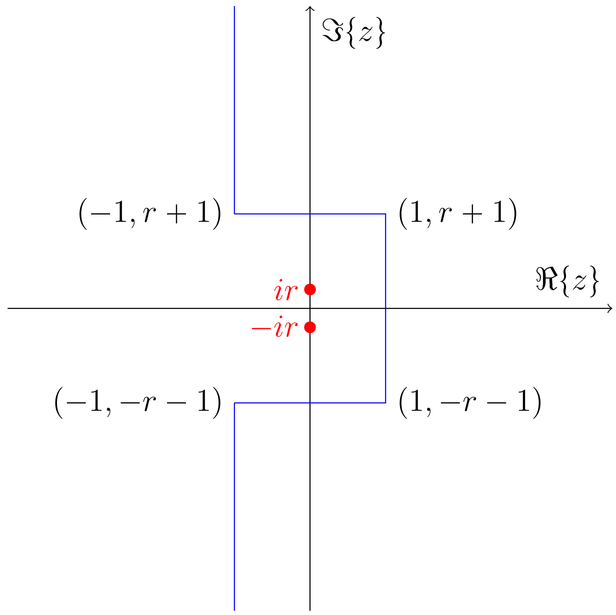


FIGURE E.1: Contour of integration

**Corollary E.0.32.** *One has*

$$k_0(x, 1/2 + ir) = \frac{1}{2\pi i} \int_{(*)} x^{-2\beta} \gamma(\beta, 1/2 + ir) \cos(\pi\beta) 2d\beta, \tag{E.11}$$

$$k_1(x, 1/2 + ir) = \frac{\sin(\pi(1/2 + ir))}{2\pi i} \int_{(0.7)} x^{-2\beta} \gamma(\beta, 1/2 + ir) 2d\beta, \tag{E.12}$$

*where the contour of integration* $(*)$ *is given on figure E.1.*

# Appendix F

## Chebyshev polynomials of the second kind

Suppose $x \in [-2, 2]$, then it can be uniquely expressed as

$$x = 2\cos\phi \text{ for } 0 \leqslant \phi \leqslant \pi. \tag{F.1}$$

For any $n \geqslant 0$, let

$$U_n(x) = e^{in\phi} + e^{i(n-2)\phi} + \ldots + e^{-in\phi} = \frac{e^{i(n+1)\phi} - e^{-i(n+1)\phi}}{e^{i\phi} - e^{-i\phi}} = \frac{\sin{(n+1)\phi}}{\sin\phi}. \tag{F.2}$$

These are polynomials in $x$ of degree $n$, called *Chebyshev polynomials of the second kind*

$$U_0(x) = 1,\ U_1(x) = x,\ U_2(x) = x^2 - 1,\ \ \ldots \tag{F.3}$$

In gerenal, $U_n(x) \in \mathbb{Z}[x]$ satisfy the recurrent relation

$$U_{n+1}(x) = xU_n(x) - U_{n-1}(x). \tag{F.4}$$

Chebyshev polynomials can be also defined in terms of the generating function

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1 - xt + t^2}. \tag{F.5}$$

For any $x \in [-2, 2]$, satisfying (F.1), we introduce the *Sato-Tate measure*

$$d\mu_{ST}(x) = \frac{2}{\pi}\sin^2\phi d\phi. \tag{F.6}$$

**Lemma F.0.33.** *Chebyshev polynomials $U_n(x)$ are orthogonal with respect to the Sato-Tate measure, i.e.*

$$\int_{\mathbb{R}} U_n(x)U_m(x)d\mu_{ST}(x) = \delta_{n,m}. \tag{F.7}$$

*Proof.* It follows from the identity below

$$\int_0^{\pi} \frac{\sin(n+1)\phi}{\sin\phi}\frac{\sin(m+1)\phi}{\sin\phi}\sin^2\phi d\phi = \frac{\pi}{2}\delta_{n,m}.$$

$\square$